Dominating sets for spaces of holomorphic functions and sampling constants

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- More generally, ν is a **sampling measure** for $\mathcal{F}$ if there exist $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{L^p(\Omega,\mu)} \overset{\substack{\text{reverse}\\ \text{Carleson}}}{\leqslant} \|f\|_{L^p(\Omega,\nu)} \overset{\text{Carleson}}{\leqslant} C_2 \|f\|_{L^p(\Omega,\mu)}, \quad \forall f \in \mathcal{F}.$$

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$$\frac{|E \cap]x-r, x+r[|}{|]x-r, x+r[|} \geqslant \gamma, \quad \forall x \in \mathbb{R} \quad \textbf{(relative density)}.$$

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- Quantification of the uncertainty principle.

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Theorem : (Luecking '81)

$E \subset \mathbb{D}$ is dominating for A^p_α iff there exist $0 < r < 1$ and $\gamma > 0$ such that

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- Green-Wagner 2021 : generalization for smooth strongly pseudoconvex domains Ω in $\mathbb{C}^n$.

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- $\phi : \mathbb{C} \to \mathbb{R}$ a subharmonic function (i.e. $\Delta\phi \geqslant 0$) such that the measure $d\mu := \Delta\phi dm$ is doubling : there exists $C > 0$, such that

$$\mu(D(z, 2r)) \leqslant C\mu(D(z, r)), \quad \forall z \in \mathbb{C}, \forall r > 0$$

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Theorem : (Jansen-Peetre-Rochberg '87, Lou-Zhuo 2019)

$E \subset \mathbb{C}$ is dominating for $\mathcal{F}^p_\phi$ iff there exist $0 < r < 1$ and $\gamma > 0$ such that

$$\frac{m(E \cap D^r(z))}{m(D^r(z))} \geqslant \gamma, \quad \forall z \in \mathbb{C} \quad \textbf{(relative density)}.$$

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If $E \subset \mathbb{D}$ is dominating for A^p_α then $C_E \lesssim \gamma^{-L(r)}$ where $L(r) = c_1 \frac{1+\alpha}{p} \frac{1}{1-r^4} \ln\left(\frac{1}{1-r}\right)$.

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Theorem : (Konate-O. 2021)

If $E \subset \mathbb{C}$ is dominating for $\mathcal{F}^p_\phi$ then $C_E \lesssim \gamma^{-L(r)}$ where $L(r) = c_1 r^{\frac{1}{\kappa}} + \frac{1}{p}(c_2 + c_3 \ln(1+r))$ (κ depends on ϕ).

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- There exist $(a_n)_{n\in\mathbb{N}}$ such that for every $r > 0$, we have $\mathbb{C} = \bigcup_{n\in\mathbb{N}} D^r(a_n)$ and

$$\sum_n \chi_{D^r(a_n)} \leqslant N \qquad \text{(non-overlapping property)}$$

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- There exist "good disks" $(D^r(a_n))_{n\in I}$ such that

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- Remez type inequality : on a "good" disk $D^r(a_n)$ we have

$$\|\phi\|_{L^p(D^r(a_n))} \lesssim \gamma^{-L(r)}\|\phi\|_{L^p(E\cap D^r(a_n))}.$$

Open questions

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- Application to control theory.

Thank you for your attention !