Integration and removable singularities for Stokes' Theorem

Besançon, 28/09/2021

Antoine Julia

Institut de Mathématiques d'Orsay

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What are the minimal assumptions ?

Applications : integration by parts and weak solutions

Extreme example :

Suppose that $u \in C^1(\mathrm{B}(0,1))$ satisfies

$$\Delta u(x) = 0 \qquad \text{for all } x \in \mathrm{B}(0,1) \setminus \mathbf{E}.$$

If, given $\phi \in C_c^\infty(\mathrm{B}(0,1))$, we can write :

$$\int_{\mathrm{B}(0,1)} \Delta\phi(x) u(x)\,\mathrm{d}x = -\int_{\mathrm{B}(0,1)} \nabla\phi \cdot \nabla u = \int_{\mathrm{B}(0,1)} \phi(x)\Delta u(x)\,\mathrm{d}x = 0.$$

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For Stokes' theorem : **minimal surfaces and calibrations**

it is necessary to allow for **singular subsets**.

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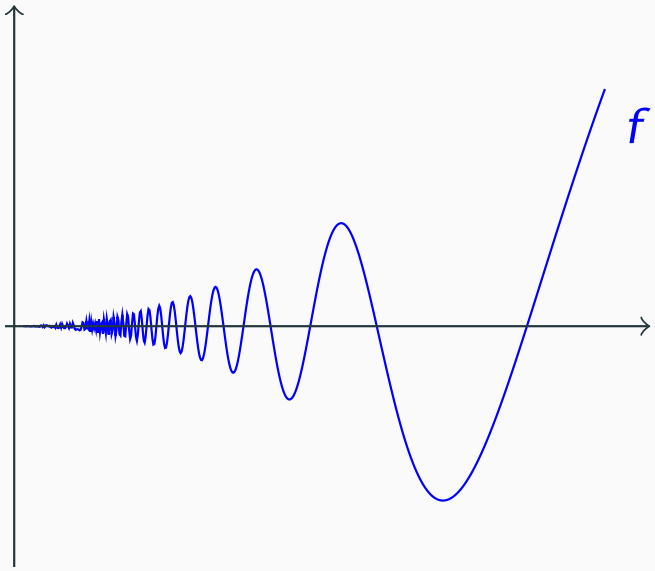

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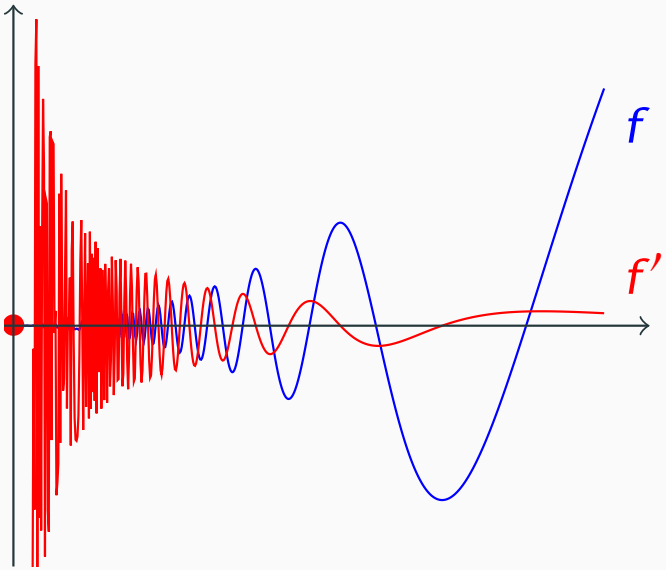

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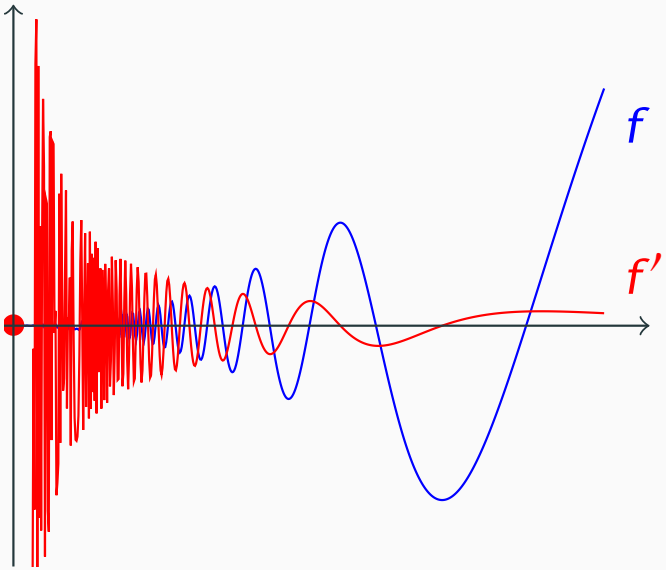

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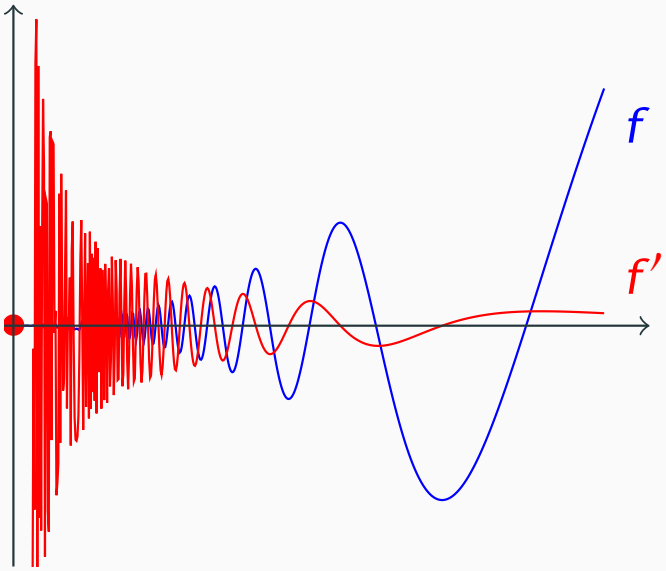

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Lebesgue is not enough,
(Denjoy, Perron, Henstock, Kurzweil).

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Definition : Riemann integral

f is **Riemann integrable** if there exists $R(f) \in \mathbb{R}$ and $\forall \varepsilon > 0$, there exists $\delta > 0$ such that for every tagged partition $([a_j, a_{j+1}], x_j)_{j=1,\dots,k}$ with $x_j \in [a_j, a_{j+1}]$ and $a_{j+1} - a_j < \delta$,

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Crucial point : a δ-fine tagged partition exists ! (Cousin's Lemma).

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Theorem. If F is differentiable, then F' is HK integrable.

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Theorem (Henstock (1961)-Kurzweil (1957))

Let $f : [0,1] \to \mathbb{R}$ be continuous,
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One could ask f to be only "pointwise Lipschitz continuous"
(and use the Rademacher-Stepanoff theorem).

Integration in higher dimension

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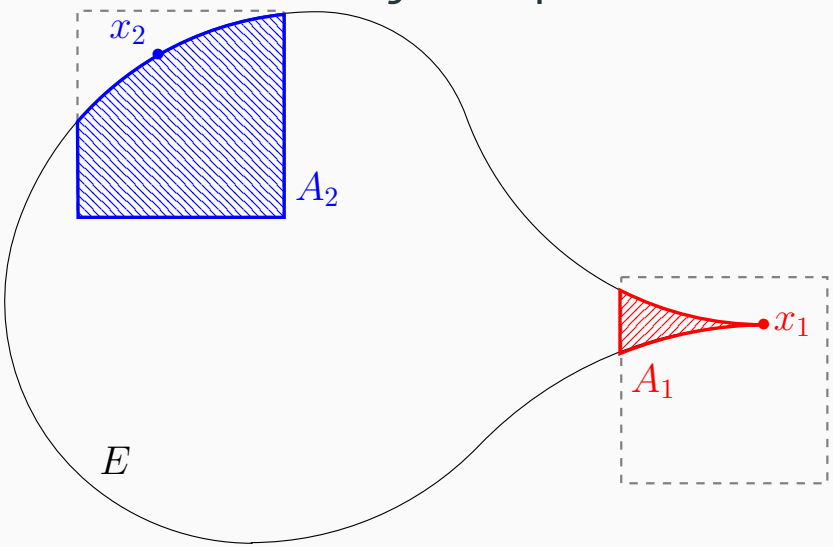

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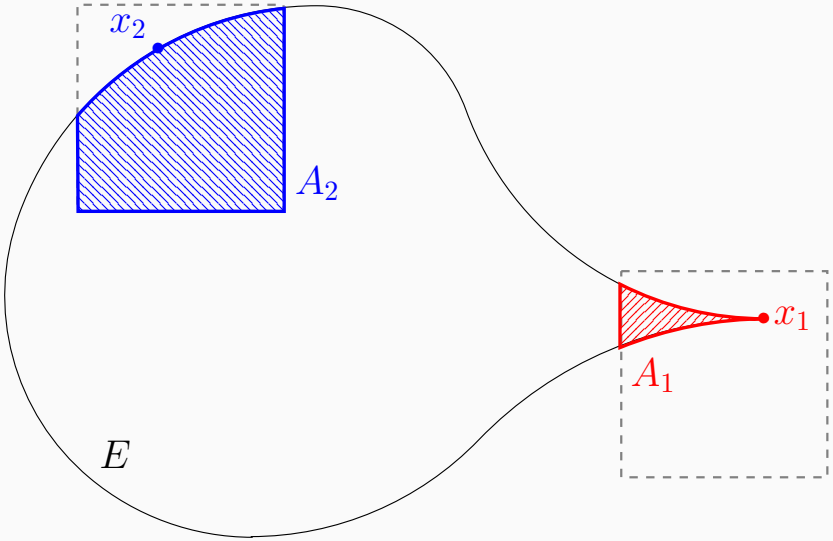

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Estimate the flux in a subset A_j

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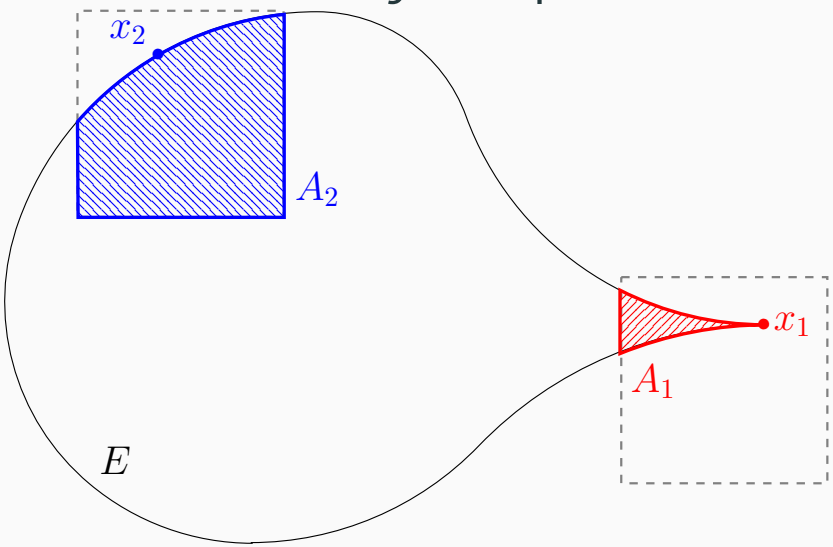

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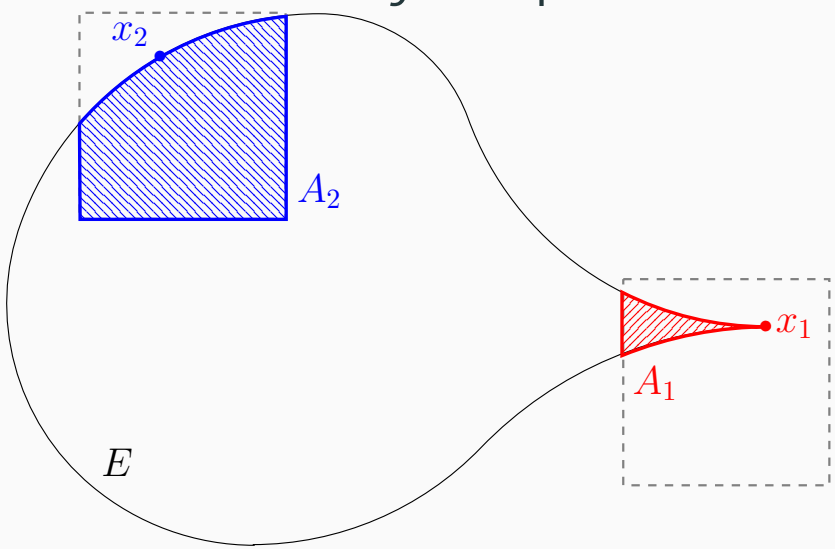

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(Mawhin, Pfeffer, Howard) : divergence theorem in BV subsets of $\mathbb{R}^m$:

For $\mathbf{v}$ continuous, differentiable except on a $\mathscr{H}^{m-1}$ σ-finite set.

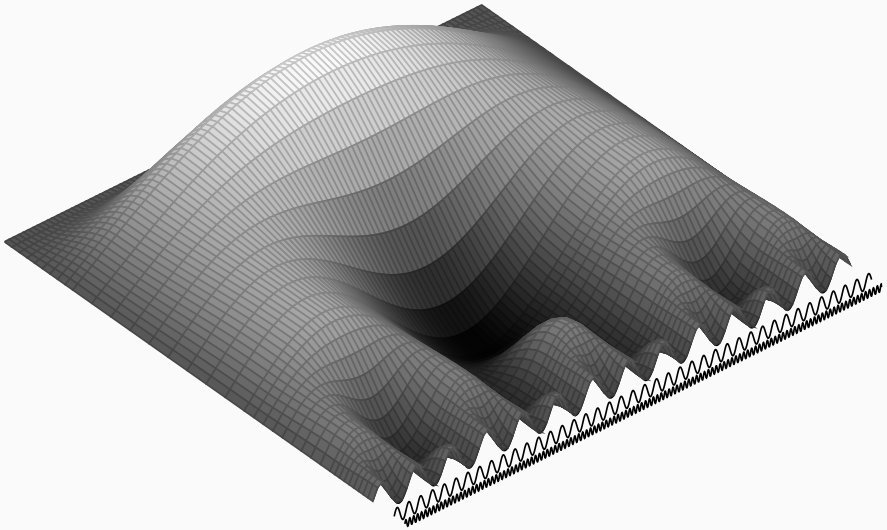
What of surfaces ?

Theorem (J., 2018)

There is a surface $M \subset \mathbb{R}^3$, with one singular point and a 1-form ω smooth except at one point such that

$$0 = \int_M \mathrm{d}\omega \neq \int_{\partial M} \omega.$$

Example with a singular segment :



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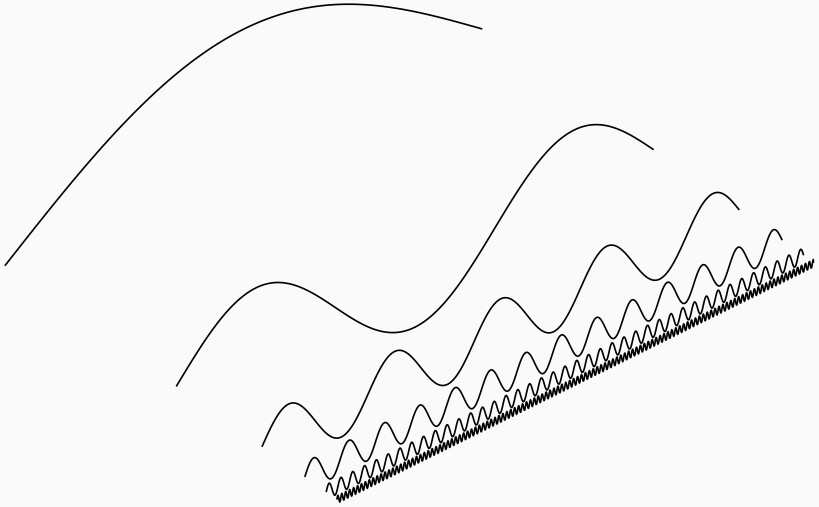

- $L(\Gamma_n) \to +\infty$,
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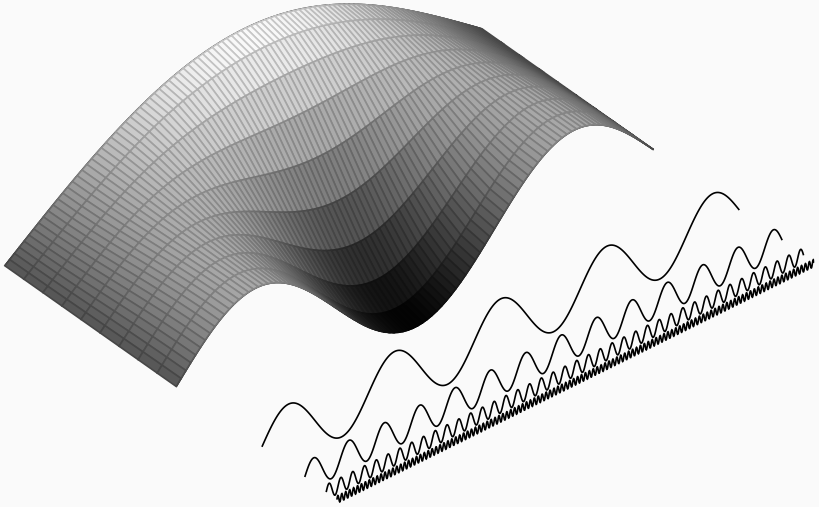

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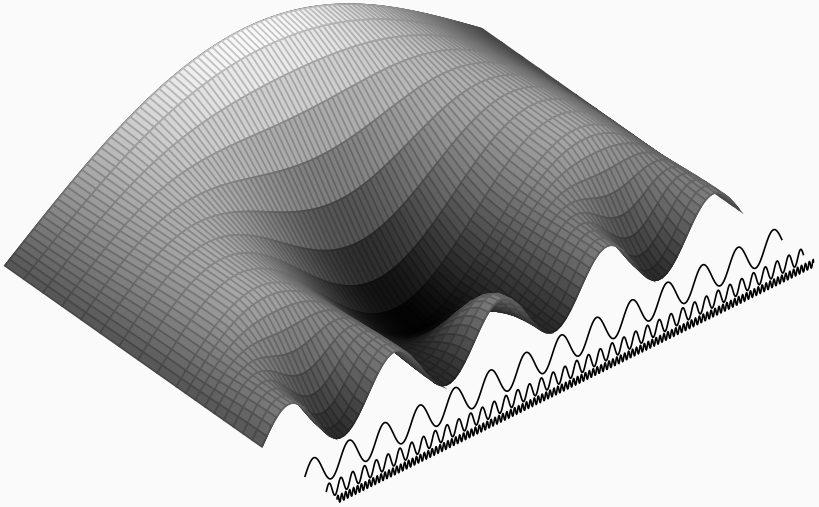

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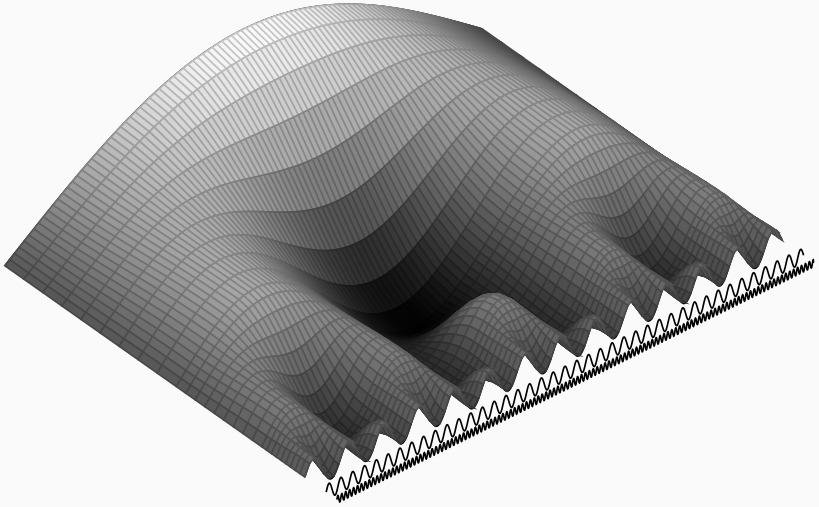

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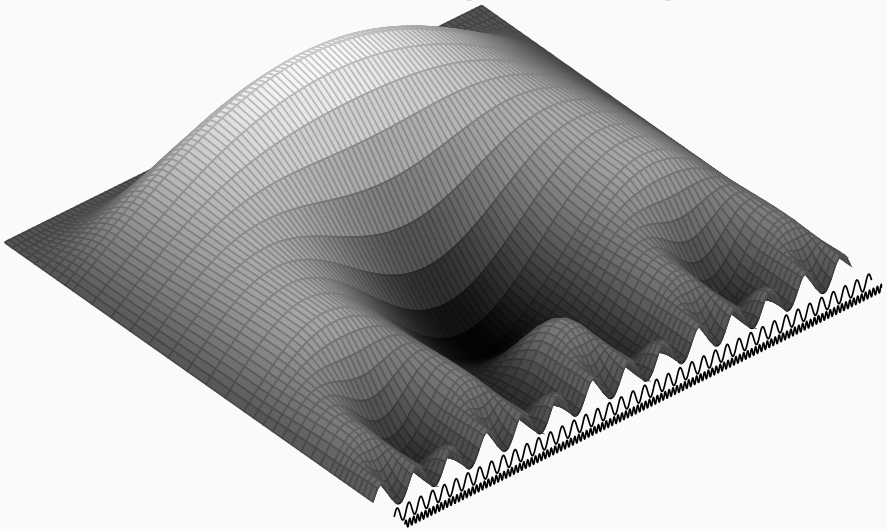

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Question :

On which singular surfaces does Stokes' Theorem hold ?

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Generalized surfaces : integral currents of dimension m in $\mathbb{R}^n$.

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Remarks

- These results have proofs in the context of Lebesgue integration.
- We can allow for discontinuities, weaker differentiability...

Questions

- Is there a "metric" way to do it ? Using Christ-David cubes ?
- Other surfaces ? Stokes' on oriented varifolds ?