Integration and removable singularities for Stokes’ Theorem

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Fundamental “theorem” of integration

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What are the minimal assumptions?
Applications: integration by parts and weak solutions

Extreme example:
Suppose that \( u \in C^1(B(0,1)) \) satisfies
\[
\Delta u(x) = 0 \quad \text{for all } x \in B(0,1) \setminus E.
\]

If, given \( \phi \in C_c^\infty(B(0,1)) \), we can write:
\[
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We say that such a set \( E \) is \textbf{removable} for \( C^1 \) harmonic functions.

General question: characterize removable sets.

Other examples:
Holomorphic functions, PDEs in divergence form, ...

For Stokes' theorem:
minimal surfaces and calibrations it is necessary to allow for singular subsets.
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1-dimensional integration

Question

$f : [0, 1] \rightarrow \mathbb{R}$, continuous, differentiable on $[0, 1] \setminus E$. Do we have

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Lebesgue is not enough,
(Denjoy, Perron, Henstock, Kurzweil).
Let us prove the fundamental theorem for a differentiable function \( F \):

**Main question**: is \( F' \) integrable?

**Definition**: \( f \) is integrable if there exists and \( \forall \varepsilon > 0 \), there exists such that for every tagged partition \( (a_j, a_j + 1, x_j) \) \( j = 1, \ldots, k \) with \( x_j \in [a_j, a_j + 1] \) and \( a_{j+1} - a_j < \delta \),

\[
\left| \left( f \right) - \sum_{j} f(x_j)(a_{j+1} - a_j) \right| < \varepsilon.
\]

\( F \) is differentiable, so Given \( \varepsilon > 0 \), there exists \( \delta > 0 \) with:

\[
|y - x| < \delta \Rightarrow |F(y) - F(x) - F'(x)(y - x)| < \varepsilon |y - x|.
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Summing over the tagged partition:

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(f) := F(1) - F(0) = \sum f(a_{j+1}) - F(a_j) = \sum f'(x_j)(a_{j+1} - a_j) + O(\varepsilon).
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**Crucial point**: a \( \delta \)-fine tagged partition exists! (Cousin's Lemma).

**Theorem**: If \( F \) is differentiable, then \( F' \) is HK integrable.
The Henstock-Kurzweil integral

Let us prove the fundamental theorem for a differentiable function $F$:

Main question: is $F'$ integrable?

**Definition: Riemann integral**

$f$ is **Riemann integrable** if there exists $R(f) \in \mathbb{R}$ and

$\forall \varepsilon > 0$, there exists $\delta > 0$ such that for every tagged partition

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$f$ is **Henstock-Kurzweil integrable** if there exists $HK(f) \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\forall x$ there exists $\delta(x) > 0$ such that for every tagged partition $([a_j, a_{j+1}], x_j)_{j=1,...,k}$ with $x_j \in [a_j, a_{j+1}]$ and $a_{j+1} - a_j < \delta(x_j)$,

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Theorem (Henstock (1961)-Kurzweil (1957))

Let \( f : [0, 1] \to \mathbb{R} \) be continuous, differentiable except on a countable set, then

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One could ask $f$ to be only “pointwise Lipschitz continuous” (and use the Rademacher-Stepanoff theorem).
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If a integral on $[0, 1]$ satisfies the generalized fundamental theorem and can be extended to $[0, 1]^2$ by Fubini then it does not satisfy the generalized divergence theorem.
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$v$ a vector field in the set

Estimate the flux in a subset $A_j$

$$\left| \text{div} \, v(x_j)|A_j| - \int_{\partial A_j} v \cdot \nu_{A_j} \right| \leq \varepsilon d(A_j) \mathcal{P}(A_j).$$
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- $x_1 \in A_1$
- $x_2 \in A_2$
- $E$

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(Mawhin, Pfeffer, Howard): divergence theorem in BV subsets of $\mathbb{R}^m$:

For $v$ continuous, differentiable except on a $\mathcal{H}^{m-1}$ $\sigma$-finite set.
What of surfaces?

Theorem (J., 2018)

There is a surface $M \subset \mathbb{R}^3$, with one singular point and a 1-form $\omega$ smooth except at one point such that

$$0 = \int_M d\omega \neq \int_{\partial M} \omega.$$

Example with a singular segment:
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- $\omega \to 0$ on $\Gamma_\infty$.
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Question:

On which singular surfaces does Stokes’ Theorem hold?
Generalized surfaces: integral currents of dimension $m$ in $\mathbb{R}^n$. 

Theorem (J., 2018)

The following currents satisfy a generalized Stokes’ Theorem:

• All $1$-dimensional integral currents (countable unions of curves)

• Mass minimizing currents with smooth boundary.

• O-minimal chains (including all compact analytic varieties).

• More generally: Currents whose singular set has finite intrinsic Minkowski content.
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- More generally: Currents whose singular set has finite intrinsic Minkowski content.
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**Remarks**

- These results have proofs in the context of Lebesgue integration.
- We can allow for discontinuities, weaker differentiability...

**Questions**

- Is there a “metric” way to do it? Using Christ-David cubes?
- Other surfaces? Stokes’ on oriented varifolds?