Traces in homogeneous Besov spaces and interpolation

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Goal, setting and motivations

Definition (Homogeneous Sobolev and Besov spaces)

Let $p, q \in [1, +\infty]$, $p \neq 1$, $s \in \mathbb{R}$, $k \in \mathbb{N}$, we define homogeneous Besov and Sobolev spaces on $\mathbb{R}^d$

$$\dot{W}^{k,p}(\mathbb{R}^d) := \{ u \in \mathcal{S}'_h(\mathbb{R}^d) \mid \|u\|_{\dot{W}^{k,p}(\mathbb{R}^d)} < +\infty \},$$

$$\dot{B}^s_{p,q}(\mathbb{R}^d) := \{ u \in \mathcal{S}'_h(\mathbb{R}^d) \mid \|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < +\infty \},$$

$$\|u\|_{\dot{W}^{k,p}(\mathbb{R}^d)} := \|\nabla^k u\|_{\mathrm{L}^p(\mathbb{R}^d)}, \ \|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} := \Big(\sum_{k\in\mathbb{Z}} 2^{qks} \|\Delta_k u\|^q_{\mathrm{L}^p(\mathbb{R}^d)}\Big)^{\frac{1}{q}}$$

where

$$\mathcal{S}'_h(\mathbb{R}^d) := \{ u \in \mathcal{S}'(\mathbb{R}^d) \mid \Theta(\lambda D)u \xrightarrow[\lambda\to+\infty]{L^\infty} 0, \forall \Theta \in C^\infty_c(\mathbb{R}^d) \}$$

with $(\Delta_j)_{j\in\mathbb{Z}}$ an homogeneous Littlewood-Paley decomposition.

Few remarks,

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- $W^{s,p}(\mathbb{R}^d_+), B^s_{p,q}(\mathbb{R}^d_+)$ have both a trace theorems $s > \frac{1}{p}$ or $q = 1$ when $s = \frac{1}{p}$,

  $$\|u\|_{W^{s-\frac{1}{p},p}(\mathbb{R}^{d-1})} \lesssim \|u\|_{W^{s,p}(\mathbb{R}^d_+)}, \quad \|u\|_{B^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{d-1})} \lesssim \|u\|_{B^s_{p,q}(\mathbb{R}^d_+)}$$

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  still true for $\dot{W}^{s,p}(\mathbb{R}^d_+), \dot{B}^s_{p,q}(\mathbb{R}^d_+)$ with above exponents conditions (completness & trace theorem).
- $\nu = -\mathfrak{e}_d$ being $\mathbb{R}^d_+$'s exterior normal, we may define

  $$\dot{B}^s_{p,q,\mathcal{D}}(\mathbb{R}^d_+) := \{\, u \in \dot{B}^s_{p,q}(\mathbb{R}^d_+) \,|\, u(\cdot,0) = 0 \text{ in } \dot{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{d-1}) \,\}$$

  $$\dot{B}^s_{p,q,\mathcal{N}}(\mathbb{R}^d_+) := \{\, u \in \dot{B}^s_{p,q}(\mathbb{R}^d_+) \,|\, \partial_{x_d} u(\cdot,0) = 0 \text{ in } \dot{B}^{s-1-\frac{1}{p}}_{p,q}(\mathbb{R}^{d-1}) \,\}.$$

  for s (resp. $s-1$) satisfying previous conditions.

Set $W^{2,p}_{\mathcal{D}} := W^{1,p}_0 \cap W^{2,p}$, $\dot{W}^{2,p}_{\mathcal{D}} := \overline{W^{2,p}_{\mathcal{D}}}^{\|\cdot\|_{\dot{W}^{2,p}}}$,
$W^{2,p}_{\mathcal{N}} = \{u \in W^{2,p} \,|\, \partial_\nu u = 0\}$, $\dot{W}^{2,p}_{\mathcal{N}} := \overline{W^{2,p}_{\mathcal{N}}}^{\|\cdot\|_{\dot{W}^{2,p}}}$.

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<u>Questions.</u>

- 1) Is it true that $(\mathrm{L}^p(\mathbb{R}^d_+), \dot{W}^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+))_{\theta,q} = \dot{B}^{2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^d_+)$?
- 2) Is there already some results in this direction ?

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- 2) Is there already some results in this direction ?
- 3) Why investigate such kind of properties ?

Question 3) : Why ? Because of a Da Prato-Grisvard theorem :

Theorem (Danchin, Hieber, Mucha, Tolksdorf - 2020)

If $X \cap D(\dot{\mathcal{A}}) = D(\mathcal{A})$, with "$D(\dot{\mathcal{A}}) = \overline{D(\mathcal{A})}^{\|\mathcal{A}\cdot\|}$".
Writing $\dot{\mathcal{D}}_{\mathcal{A}}(\theta, q) = (X, D(\dot{\mathcal{A}}))_{\theta,q}$ for $(\theta, q) \in (0,1) \times [1, +\infty)$, for $q \in [1, +\infty)$, $\theta \in (0, \frac{1}{q})$, and $T \in (0, +\infty]$.
For any $f \in \mathrm{L}^q(0, T; \mathcal{D}_{\mathcal{A}}(\theta, q))$, $u_0 \in \dot{\mathcal{D}}_{\mathcal{A}}(\theta_q, q)$,
$\exists! u \in C^0([0, T), \dot{\mathcal{D}}_{\mathcal{A}}(\theta_q, q))$, satisfying

$$\partial_t u + \mathcal{A}u = f \text{ on } (0, T), \text{ and } u(0, \cdot) = u_0$$

$\exists C(\theta, q, \mathcal{A}) > 0$, such that

$$\|u\|_{\mathrm{L}^\infty(0,T;\dot{\mathcal{D}}_{\mathcal{A}}(\theta_q,q))} + \|\partial_t u\|_{\mathrm{L}^q(0,T;\dot{\mathcal{D}}_{\mathcal{A}}(\theta,q))} + \|\mathcal{A}u\|_{\mathrm{L}^q(0,T;\dot{\mathcal{D}}_{\mathcal{A}}(\theta,q))} \leqslant C\left(\|f\|_{\mathrm{L}^q(0,T;\dot{\mathcal{D}}_{\mathcal{A}}(\theta,q))} + \|u_0\|_{\dot{\mathcal{D}}_{\mathcal{A}}(\theta_q,q)}\right)$$

Question 2) : Already few results ? Yes, see Davide Guidetti 1991.

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Few remarks :

- 1) Only for inhomogeneous Besov spaces $B^s_{p,q}(\Omega)$.
- 2) For Ω, to be $\mathbb{R}^d_+$, or a smooth bounded domain.
- 3) Very general boundary conditions with smooth coefficients (Lopatinskii-Shapiro boundary conditions).

Guidetti's strategy : build elliptic regularity/resolvent estimates involving the desired boundary conditions in Besov spaces to construct pre-image.

Its general result gives obviously the inhomogeneous counterpart of the desired result :

Theorem (Guidetti - 1991)

For $\mathcal{J} \in \{\mathcal{D}, \mathcal{N}\}$, $p \in (1, +\infty)$, $\theta \in (0, 1)$, $q \in [1, +\infty]$,

$$(\mathrm{L}^p(\mathbb{R}^d_+), W^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+))_{\theta,q} = B^{2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^d_+).$$

Question 1) Is $(\mathrm{L}^p(\mathbb{R}^d_+), \dot{W}^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+))_{\theta,q} = \dot{B}^{2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^d_+)$? Yes, up to density argument.

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- Indeed, from above theorem, and obvious embeddings we get

$$\begin{aligned}(\mathrm{L}^p(\mathbb{R}^d_+), W^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+))_{\theta,q} &\hookrightarrow (\mathrm{L}^p(\mathbb{R}^d_+), \dot{W}^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+))_{\theta,q} \\ &\hookrightarrow (\mathrm{L}^p(\mathbb{R}^d_+), \dot{W}^{2,p}(\mathbb{R}^d_+))_{\theta,q} = \dot{B}^{2\theta}_{p,q}(\mathbb{R}^d_+),\end{aligned}$$

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so by dilation argument

$$\|u\|_{\dot{B}^{2\theta}_{p,q}(\mathbb{R}^d_+)} \leqslant \|u\|_{(\mathrm{L}^p(\mathbb{R}^d_+),\dot{W}^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+)))_{\theta,q}} \lesssim \|u\|_{\dot{B}^{2\theta}_{p,q}(\mathbb{R}^d_+)}.$$

$\forall u \in \mathrm{L}^p(\mathbb{R}^d_+) \cap \dot{W}^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+) = W^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+)$. Hence it suffices to proves that $W^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+)$ is dense in $\dot{B}^{2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^d_+)$ when the latter is complete.

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Laplacians in homogeneous Besov spaces

Proposition (G. 2021)

Let $p \in (1, +\infty)$, $q \in [1, +\infty)$, $s \in (-\frac{1}{p'}, \frac{1}{p})$, $\nu \in (0, \frac{\pi}{2})$, *for all* $\lambda \in \Sigma_\nu$, *for any* $f \in \dot{B}^s_{p,q}(\mathbb{R}^d_+)$, *there exists a unique* $u \in \dot{B}^s_{p,q}(\mathbb{R}^d_+) \cap \dot{B}^{s+2}_{p,q,\mathcal{J}}(\mathbb{R}^d_+)$ *such that*

$$\lambda u - \Delta_{\mathcal{J}} u = f \text{ in } \mathbb{R}^d_+,$$

with estimate

$$|\lambda| \, \|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^d_+)} + |\lambda|^{\frac{1}{2}} \, \|\nabla u\|_{\dot{B}^s_{p,q}(\mathbb{R}^d_+)} + \left\|\nabla^2 u\right\|_{\dot{B}^s_{p,q}(\mathbb{R}^d_+)} \lesssim \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d_+)} .$$

$\{(\lambda I - \Delta_{\mathcal{J}})^{-1}\}_{\lambda \in \Sigma_\nu}$ *is a family of Banach isomorphisms from* $\dot{B}^s_{p,q}(\mathbb{R}^d_+)$ *to* $\dot{B}^s_{p,q}(\mathbb{R}^d_+) \cap \dot{B}^{s+2}_{p,q,\mathcal{J}}(\mathbb{R}^d_+)$.

Theorem (G. 2021)

Let $\mathcal{J} \in \{\mathcal{D}, \mathcal{N}\}$, for $p \in (1, +\infty)$, $q \in [1, +\infty)$, $s \in \mathbb{R}$, the operator

$$-\Delta_{\mathcal{J}} : \dot{B}^{s}_{p,q,\mathcal{J}}(\mathbb{R}^d_+) \longrightarrow \dot{B}^{s-2}_{p,q}(\mathbb{R}^d_+)$$

is an isomorphism of Banach spaces whenever

- $\mathcal{J} = \mathcal{D}$, $\frac{1}{p} < s < \frac{d}{p}$ *or* $s = \frac{1}{p}, \frac{d}{p}$ *and* $q = 1$,
- $\mathcal{J} = \mathcal{N}$, $1 + \frac{1}{p} < s < \frac{d}{p}$ *or* $s = 1 + \frac{1}{p}, \frac{d}{p}$ *and* $q = 1$.

Density and Interpolation results

Proposition (G. 2021)

Let $p \in (1, +\infty)$, $q \in [1, +\infty)$, $s \in (0, 2)$, *we have that*

$$\overline{W^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+)}^{\|\cdot\|_{\dot{B}^s_{p,q}(\mathbb{R}^d_+)}} = \dot{B}^s_{p,q,\mathcal{J}}(\mathbb{R}^d_+)$$

whenever

- $\mathcal{J} = \mathcal{D}$, $\frac{1}{p} < s < \frac{d}{p}$ *or* $s = \frac{1}{p}, \frac{d}{p}$ *and* $q = 1$,
- $\mathcal{J} = \mathcal{N}$, $1 + \frac{1}{p} < s < \frac{d}{p}$ *or* $s = 1 + \frac{1}{p}, \frac{d}{p}$ *and* $q = 1$.

Hence

Theorem (G. 2021)

For $\mathcal{J} \in \{\mathcal{D}, \mathcal{N}\}$, $p \in (1, +\infty)$, $\theta \in (0, 1)$, $q \in [1, +\infty]$, with either

- $\theta \in (0, \frac{d}{2p})$, and $q \in (1, +\infty)$,
- $\theta \in (0, \frac{d}{2p}]$, and $q = 1$,

$$(\mathrm{L}^p(\mathbb{R}^d_+), \dot{W}^{2,p}_{\mathcal{J}}(\mathbb{R}^d_+))_{\theta,q} = \dot{B}^{2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^d_+).$$

An application with the Hodge Laplacian

The particular case when $d = 3$ on 1-forms (vector fields). For $u \in W^{2,p}(\mathbb{R}^3_+, \mathbb{C}^3)$,

$$-\Delta_{\mathcal{H}} u = \operatorname{curl} \operatorname{curl} u - \nabla \operatorname{div} u,$$

with BC

$$[u \cdot \nu = 0,\ \nu \times \operatorname{curl} u = 0] \text{ or } [u \times \nu = 0,\ (\operatorname{div} u)\nu = 0].$$

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In this case (restricted to vector fields),

$$(\mathrm{L}^p, \mathrm{D}_p(\dot{\Delta}_{\mathcal{H}}))_{\theta,q} = [\dot{B}^{2\theta}_{p,q,\mathcal{D}} \times (\dot{B}^{2\theta}_{p,q,\mathcal{N}})^2] \text{ or } [(\dot{B}^{2\theta}_{p,q,\mathcal{D}})^2 \times (\dot{B}^{2\theta}_{p,q,\mathcal{N}})].$$

Extension to differential forms

Let $\Lambda := \Lambda_{\mathbb{C}}\mathbb{R} \simeq \mathbb{C}^{2^d}$ being the complex exterior algebra over $\mathbb{R}$, we define

- $X(\mathbb{R}^d_+, \Lambda)$ for differentials forms whose coefficients are in $X(\mathbb{R}^d_+)$, $X \in \{\dot{W}^{s,p}, \dot{B}^s_{p,q}\}$.
- $\wedge$ and $\lrcorner$ are respectively the exterior and interior product.
- d and δ stand for the exterior and interior derivatives.

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- d and δ stand for the exterior and interior derivatives.

• All previous results remain true for boundary conditions $u \wedge \nu = 0$ and $\nu \lrcorner u = 0$, due to symmetry properties in $\mathbb{R}^d_+$:

$$\nu \wedge u = 0 \iff 2^{d-1} \text{ scalar coordinates } u_I(\cdot, 0) = 0,$$
$$\nu \lrcorner u = 0 \iff 2^{d-1} \text{ scalar coordinates } u_{I'}(\cdot, 0) = 0.$$

it reduces to scalar case.

- The same goes with both type Hodge boundary conditions :

$$\begin{aligned}
\nu \wedge u = 0, \nu \lrcorner \, \mathrm{d}u = 0 \iff & 2^{d-1} \text{ scalar coordinates } u_I(\cdot, 0) = 0, \\
& 2^{d-1} \text{ scalar coordinates } \partial_{x_d} u_{I'}(\cdot, 0) = 0, \\
\nu \lrcorner \, u = 0, \nu \wedge \delta u = 0 \iff & 2^{d-1} \text{ scalar coordinates } u_{I'}(\cdot, 0) = 0, \\
& 2^{d-1} \text{ scalar coordinates } \partial_{x_d} u_I(\cdot, 0) = 0.
\end{aligned}$$

- Application : the Hodge Laplacian $-\Delta_{\mathcal{H}} = \mathrm{d}\delta + \delta\mathrm{d}$ with one of above BC. Its L^p-domain is $\mathrm{D}_p(\Delta_{\mathcal{H}}) = (W^{2,p}_{\mathcal{D}})^{2^{d-1}} \times (W^{2,p}_{\mathcal{N}})^{2^{d-1}}$, with homogeneous interpolation spaces

$$(\mathrm{L}^p, \mathrm{D}_p(\dot{\Delta}_{\mathcal{H}}))_{\theta,q} = (\dot{B}^{2\theta}_{p,q,\mathcal{D}})^{2^{d-1}} \times (\dot{B}^{2\theta}_{p,q,\mathcal{N}})^{2^{d-1}}.$$

Thank you for your attention.