Volume product, polytopes and finite dimensional Lipschitz-free spaces.

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Mainly based on

Matthew Alexander, M. F. and Artem Zvavitch.
Polytopes of Maximal Volume Product.
Discrete and Computational Geometry 62 (3) (2019)

Matthew Alexander, M. F., Luis C. García-Lirola and Artem Zvavitch.
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Blaschke-Santaló inequality

The polar and the volume product of a symmetric convex body $K \subset \mathbb{R}^n$ are $K^\circ = \{y \in \mathbb{R}^n; \langle y, x \rangle \leq 1, \forall\, x \in K\}$ and $\mathcal{P}(K) = |K||K^\circ|$. The volume product is invariant with respect to inversible linear transform.

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with equality if and only if K is an ellipsoid.

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- Maximum for polytopes with fixed number of vertices: Meyer–Reisner (2011) for $n = 2$ and Alexander–Fradelizi–Zvavitch (2019) for $n \ge 3$.

Mahler's conjecture and Hanner polytopes

Mahler's conjecture, symmetric case

$$\mathcal{P}(K) \geq \mathcal{P}([-1,1]^n) = \frac{4^n}{n!},$$

with equality if and only if K is a Hanner polytope.

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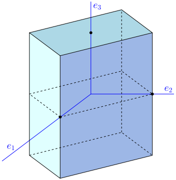
For $K \subset \mathbb{R}^{n_1}$, $L \subset \mathbb{R}^{n_2}$ with $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$, we construct in $\mathbb{R}^n$:

- $K \oplus_\infty L = K + L$ the ℓ_∞-sum: $\|(x_1, x_2)\|_{K \oplus_\infty L} = \max\{\|x_1\|_K, \|x_2\|_L\}$
- $K \oplus_1 L = \mathrm{conv}(K \cup L)$ their ℓ_1-sum: $\|(x_1, x_2)\|_{K \oplus_1 L} = \|x_1\|_K + \|x_2\|_L$



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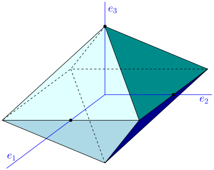

$(I \oplus_\infty I) \oplus_1 I$

A **Hanner polytope** is the iterated ℓ_1 or ℓ_∞ sum of segments.
If $K \subset \mathbb{R}^n$ is a Hanner polytope, then $\mathcal{P}(K) = \frac{4^n}{n!}$.

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- Polytopes with few vertices: (Lopez–Reisner 1998), (Meyer–Reisner 2006).
- Functional forms: (Klartag–Milman 2005), (F.–Meyer 2008-10), (Gordon–F.–Meyer–Reisner 2010), (F. Nakhle IMRN 2021+).

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- A body with a point of positive curvature is not a minimizer: (Stancu, 2009), (Reisner–Schütt–Werner 2010), (Gordon–Meyer 2011).
- Close to Hanner polytopes/Unconditional bodies: (Nazarov–Petrov–Ryabogin–Zvavitch 2010), (Kim 2013), (Kim–Zvavitch 2013).
- bodies with 'many' symmetries: (Barthe–Fradelizi 2010), (Iriyeh–Shibata 2021+).
- It follows from Viterbo's conjecture in symplectic geometry, (Artstein-Avidan–Karasev–Ostrover 2014).
- Hyperplane sections of ℓ_p-balls and Hanner polytopes, (Karasev 2019).

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Shadow systems

- Definition (Shephard '64): $(K_t)_t$ is a shadow system if $K_t = P_t(C)$ is the projection on $\mathbb{R}^n$ parallel to $e_{n+1} - t\theta$ of a closed convex set C in $\mathbb{R}^{n+1}$, where $\theta \in S^{n-1}$ is fixed.

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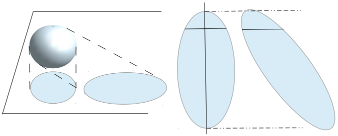

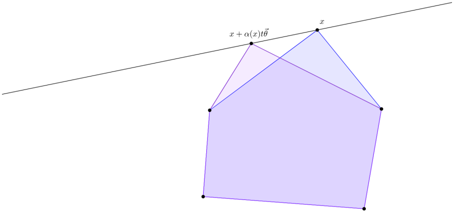

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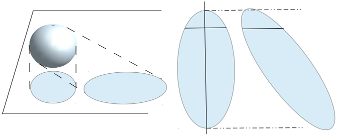

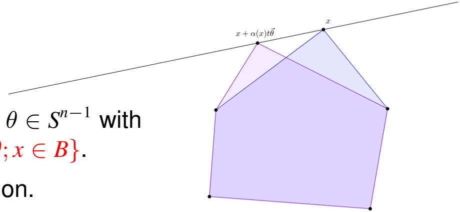

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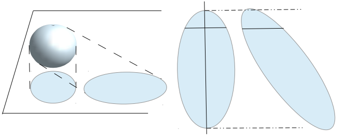

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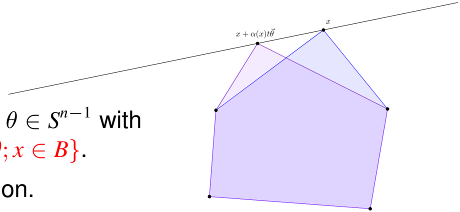

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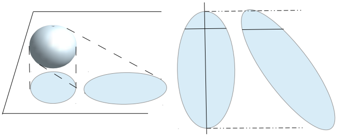

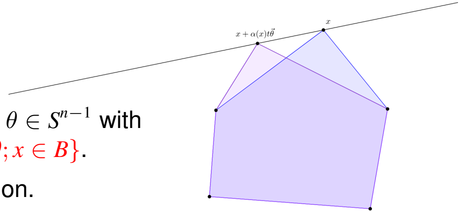

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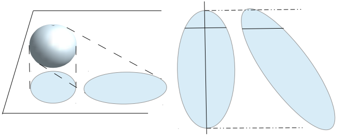

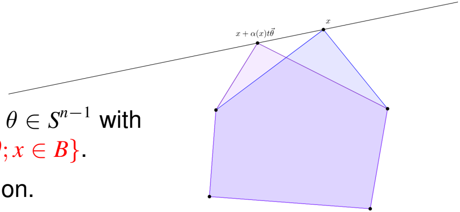

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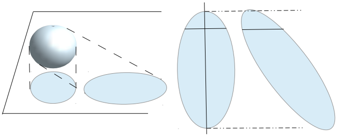

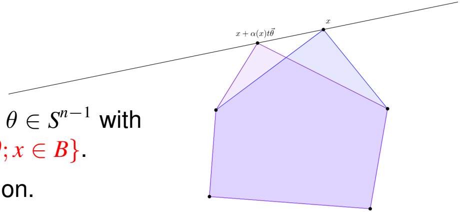

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- Cordero-Erausquin-F.-Paouris-Pivovarov '15: Generalization of Campi-Gronchi to more measures.

Shadow systems of polytopes

Let $n \geq 2$ and $m \geq n+1$. Let K be a polytope with at most m vertices. Let $0 \leq k \leq n-1$ and F be a k-face of K. Let x in the relative interior of F and u in the normal cone. Let $K_t = \mathrm{conv}(K, x+tu)$, $t > 0$.

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- if $n = 2$ then a maximizer of these moves is an affine image of a regular polygon. Meyer-Reisner (2011) + Alexander-F.-Zvavitch (2019).
- a simplicial minimizers of these moves is an affine image of an ℓ_1 ball F.-Meyer-Zvavitch (2012).
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3) $1 \leq k \leq n-1$: Alexander-F.-García-Lirola-Zvavitch (2021): generalization of the case $k = n-1$ and application to the maximizer of the volume product among Lipschitz-free balls. See below.

A local maximizer is simplicial

Theorem (Alexander–F.–Zvavitch, 2019)

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Thus for any facet F one has $|\mathcal{E}(F)| = n$. Thus F is a simplex and so K is simplicial.

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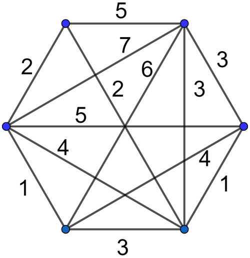

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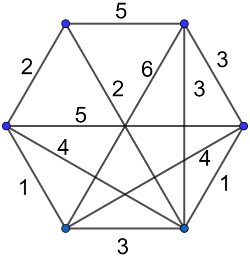

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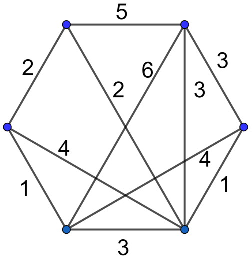

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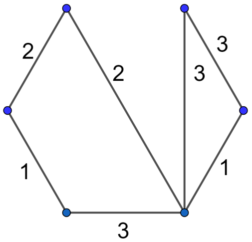

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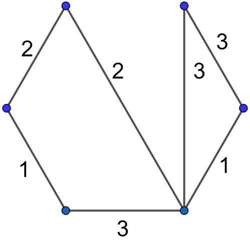

- (a_i, a_j) is an edge of the graph if and only if $d(a_i, a_j) < d(a_i, z) + d(z, a_j)$ for all $z \in M \setminus \{a_i, a_j\}$.

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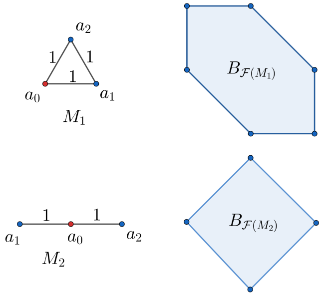
$B_{\mathcal{F}(M)}$ is a symmetric convex body of $\mathbb{R}^n$, a polytope having at most $n(n+1)$ vertices, called alcoved polyhedron, polytrope. $\mathcal{F}(M)$ is also called Arens-Eells, Wasserstein W_1, Kantorovich-Rubinstein, ...



When $\mathcal{F}(M)$ and $\mathcal{F}(M')$ are isometric?

- To (M, d) we associate a weighted graph $G = (V, E, d)$.
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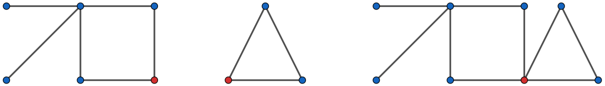

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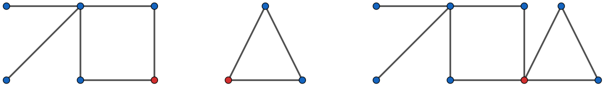

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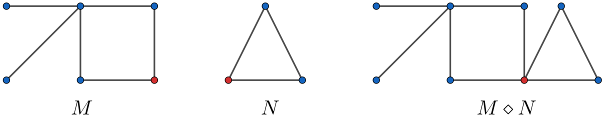

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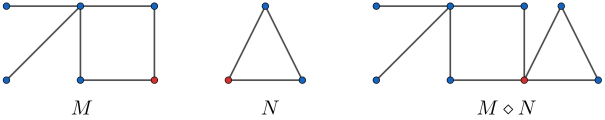

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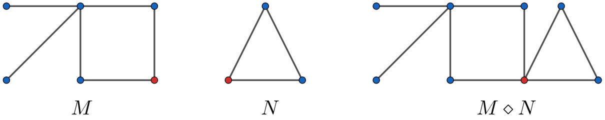

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Theorem (Alexander–F.–García-Lirola–Zvavitch, 2021)

$\mathcal{F}(M)$ and $\mathcal{F}(M')$ are isometric if and only if $|M| = |M'|$ and there exists a cyclic bijection $\sigma : E \to E'$ such that $e \mapsto d(\sigma(e))/d(e)$ is constant on each 2-connected component of G.

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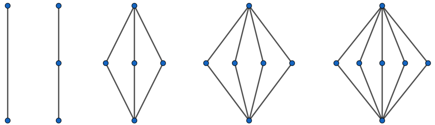
2) $B_{\mathcal{F}(M)}$ is a Hanner polytope if and only if the 2-connected components of M are bipartite graphs $K_{2,m}$, with constant weight.



$$B_{K_{2,m}} = B_1^m \oplus_\infty [-1, 1]$$

Maximizers of the volume product of a metric space

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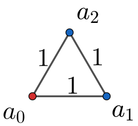

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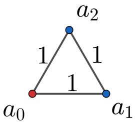

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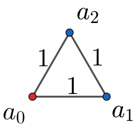
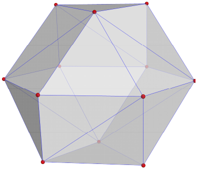


Theorem (Alexander–F.–García-Lirola–Zvavitch, 2021)

Let M be a metric space with $n+1$ points which maximizes $\mathcal{P}(M)$ among the metric spaces with $n+1$ points. Then

- *$B_{F(M)}$ has $n(n+1)$ vertices: the associated graph is K_{n+1}.*
- *$B_{\mathcal{F}(M)}$ is simplicial: all its facets are simplices.*

If $n \geq 3$ and M is K_{n+1} with equal weights, then $B_{\mathcal{F}(M)}$ is not simplicial!
Therefore it doesn't maximize the volume product.



Thank you!