On Universal Harmonic Functions on Trees

Nikiforos Biehler

Université Gustave Eiffel

Journées du GDR AFHP,
Besançon,
28 September 2021

Universality

In the most general setting, the following definition of universality has been suggested:

Definition

Let X, Y be topological spaces and $T_i : X \to Y$ $(i \in I)$ a family of continuous mappings. An element x is called universal for the family $\{T_i\}_{i \in I}$ if the set $\{T_i(x) \,|\, i \in I\}$ is dense in Y.

We observe universal behaviour in many function spaces. The first instance of universality was recorded by M. Fekete in 1914, and since then many more examples of universal objects have been found.

Universality

A typical example is the following, given by F. Bayart. We need the following definition:

Definition

Let $\omega : [0,1) \to [0,+\infty)$ be a growth rate ($\omega(r) \uparrow +\infty$). The space $H_\omega(\mathbb{D})$ is the space of holomorphic functions $f(z)$ defined on the unit disc $\mathbb{D} \subset \mathbb{C}$ that satisfy:

- $\sup_{z\in\mathbb{D}} \frac{|f(z)|}{\omega(|z|)} < \infty$;
- $\lim_{r\to 1} \max_{|z|=r} \frac{|f(z)|}{\omega(|z|)} = 0$.

Universality

A typical example is the following, given by F. Bayart. We need the following definition:

Definition

Let $\omega : [0, 1) \to [0, +\infty)$ be a growth rate $(\omega(r) \uparrow +\infty)$. The space $H_\omega(\mathbb{D})$ is the space of holomorphic functions $f(z)$ defined on the unit disc $\mathbb{D} \subset \mathbb{C}$ that satisfy:

- $\sup_{z\in\mathbb{D}} \frac{|f(z)|}{\omega(|z|)} < \infty$;
- $\lim_{r\to 1} \max_{|z|=r} \frac{|f(z)|}{\omega(|z|)} = 0$.

Universal Radial Limits, Bayart, 2004

The set of functions $f \in H_\omega(\mathbb{D})$ such that, given any measurable function ϕ on $\partial\mathbb{D}$, there exists a sequence $0 < r_j < r_{j+1} < 1 \quad \forall j$ with $\lim_{j\to\infty} r_j = 1$ with:
$\lim_{j\to\infty} f(r_j\xi) = \phi(\xi)$, for almost every $\xi \in \partial\mathbb{D}$,
is G_δ-dense in $H_\omega(\mathbb{D})$.

Notation and Basic Definitions

Tree

A connected, locally finite, countable graph,without non-trivial loops. We shall denote both the tree and it's vertex set by T.

- For each $x, y \in T$ there exists a unique path $\{z_0, z_1, \ldots, z_n\}$ of length n such that $z_0 = x, z_n = y$ and $z_k \sim z_{k+1} \quad \forall\, k < n$. Call it geodesic path $[x, y]$. The length n is a metric on T.

Notation and Basic Definitions

Tree

A connected, locally finite, countable graph,without non-trivial loops. We shall denote both the tree and it's vertex set by T.

- For each $x, y \in T$ there exists a unique path $\{z_0, z_1, \ldots, z_n\}$ of length n such that $z_0 = x, z_n = y$ and $z_k \sim z_{k+1} \quad \forall\, k < n$. Call it geodesic path $[x, y]$. The length n is a metric on T.
- Fix a reference vertex $o \in T$ called the origin. We obtain a partial ordering: $x \leq y \Leftrightarrow x \in [o, y]$.

Notation and Basic Definitions

Tree

A connected, locally finite, countable graph,without non-trivial loops. We shall denote both the tree and it's vertex set by T.

- For each $x, y \in T$ there exists a unique path $\{z_0, z_1, \dots, z_n\}$ of length n such that $z_0 = x, z_n = y$ and $z_k \sim z_{k+1} \quad \forall\, k < n$. Call it geodesic path $[x, y]$. The length n is a metric on T.
- Fix a reference vertex $o \in T$ called the origin. We obtain a partial ordering: $x \leq y \Leftrightarrow x \in [o, y]$.
- Each $x \neq o$ has a unique neighbour $y \geq x$, called the parent of x, and it is denoted by x^-. We shall assume that x has at least two other neighbours, which we call the children.

Notation and Basic Definitions

Tree

A connected, locally finite, countable graph,without non-trivial loops. We shall denote both the tree and it's vertex set by T.

- For each $x, y \in T$ there exists a unique path $\{z_0, z_1, \dots, z_n\}$ of length n such that $z_0 = x, z_n = y$ and $z_k \sim z_{k+1} \quad \forall\, k < n$. Call it geodesic path $[x, y]$. The length n is a metric on T.
- Fix a reference vertex $o \in T$ called the origin. We obtain a partial ordering: $x \leq y \Leftrightarrow x \in [o, y]$.
- Each $x \neq o$ has a unique neighbour $y \geq x$, called the parent of x, and it is denoted by x^-. We shall assume that x has at least two other neighbours, which we call the children.
- Denote be T_n the set of all vertices at distance n from the origin. T_n can be thought of as the circle of radius n, centred at the origin. We call that the n-level of the tree.

Notation and Basic Definitions

The space $\mathbb{C}^T$, of complex valued functions on the tree, is equipped with the product topology, which can be given by the following metric:

$$\rho(f,g) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{|f(x_j) - g(x_j)|}{1 + |f(x_j) - g(x_j)|},$$

where $\{x_j\}_{j=0}^{\infty}$ is an arbitrary, but fixed, enumeration of the tree.

- Topology is the same regardless the enumeration.

Notation and Basic Definitions

The space $\mathbb{C}^T$, of complex valued functions on the tree, is equipped with the product topology, which can be given by the following metric:

$$\rho(f,g) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{|f(x_j) - g(x_j)|}{1 + |f(x_j) - g(x_j)|},$$

where $\{x_j\}_{j=0}^{\infty}$ is an arbitrary, but fixed, enumeration of the tree.

- Topology is the same regardless the enumeration.
- $(\mathbb{C}^T, \rho)$ is a complete metric space, and metric is equivalent to pointwise convergence.

Notation and Basic Definitions

Boundary of the Tree

The boundary is defined as the set of all infinite geodesic paths $e = (o = e_0, e_1, \ldots, e_n \ldots)$ starting from the origin o. It is denoted by ∂T.

- For $x \in T$ define the boundary sector $B_x \subset \partial T$ as the set of all $e \in \partial T$ s.t. $x \in e$.

Notation and Basic Definitions

Boundary of the Tree

The boundary is defined as the set of all infinite geodesic paths $e = (o = e_0, e_1, \dots, e_n \dots)$ starting from the origin o. It is denoted by ∂T.

- For $x \in T$ define the boundary sector $B_x \subset \partial T$ as the set of all $e \in \partial T$ s.t. $x \in e$.
- The sets $\{B_x\}_{x \in T}$ form the open basis of a topology on ∂T. With that topology ∂T is compact and totally disconnected.

Notation and Basic Definitions

Boundary of the Tree

The boundary is defined as the set of all infinite geodesic paths $e = (o = e_0, e_1, \ldots, e_n \ldots)$ starting from the origin o. It is denoted by ∂T.

- For $x \in T$ define the boundary sector $B_x \subset \partial T$ as the set of all $e \in \partial T$ s.t. $x \in e$.
- The sets $\{B_x\}_{x \in T}$ form the open basis of a topology on ∂T. With that topology ∂T is compact and totally disconnected.
- The sets $\{B_x\}_{x \in T_n}$ generate a σ-algebra, $\mathcal{M}_n$ on ∂T. That sequence is nested: $\mathcal{M}_n \subset \mathcal{M}_{n+1}$

Notation and Basic Definitions

Boundary of the Tree

The boundary is defined as the set of all infinite geodesic paths $e = (o = e_0, e_1, \dots, e_n \dots)$ starting from the origin o. It is denoted by ∂T.

- For $x \in T$ define the boundary sector $B_x \subset \partial T$ as the set of all $e \in \partial T$ s.t. $x \in e$.
- The sets $\{B_x\}_{x \in T}$ form the open basis of a topology on ∂T. With that topology ∂T is compact and totally disconnected.
- The sets $\{B_x\}_{x \in T_n}$ generate a σ-algebra, $\mathcal{M}_n$ on ∂T. That sequence is nested: $\mathcal{M}_n \subset \mathcal{M}_{n+1}$
- These σ-algebras $\mathcal{M}_n$, generate the Borel σ-algebra on ∂T, namely $\mathcal{M} := \sigma(\bigcup_{n=0}^{\infty} \mathcal{M}_n)$.

Harmonic functions and Martingales

Consider a set of transition probabilities Q on the set of neighbours of the tree that satisfies:

$$\sum_{y \sim x} q(x,y) = 1 \quad \forall\, x, y \in T,\; q(x,y) > 0 \textit{ for } y \geq x \textit{ and } q(x, x^-) = 0.$$

A function $f : T \to \mathbb{C}$ is said to be Q-harmonic if it satisfies:

$$\sum_{y \sim x} q(x,y) f(y) = f(x)\,, \quad \forall x \in T.$$

The set of all these functions is denoted by $H_Q(T)$ and it is a closed linear subspace of $\mathbb{C}^T$, therefore a complete metric space.

Harmonic functions and Martingales

Such an operator induces a probability measure on the boundary of the tree in the following way:

- Define the probability of B_x as $p(B_x) = \prod_{k=0}^{n-1} q(x_k, x_{k+1})$ where $x_0 = o, x_n = x$ and $x_k \leq x_{k+1}$.

Harmonic functions and Martingales

Such an operator induces a probability measure on the boundary of the tree in the following way:

- Define the probability of B_x as $p(B_x) = \prod_{k=0}^{n-1} q(x_k, x_{k+1})$ where $x_0 = o, x_n = x$ and $x_k \leq x_{k+1}$.
- Extend these to measures $\mathbb{P}_n$ on $\mathcal{M}_n$ by using the fact that $\{B_x\}_{x \in T_n}$ is a finite partition of $\mathcal{M}_n$ and $\sum_{y \sim x} q(x, y) = 1$.

Harmonic functions and Martingales

Such an operator induces a probability measure on the boundary of the tree in the following way:

- Define the probability of B_x as $p(B_x) = \prod_{k=0}^{n-1} q(x_k, x_{k+1})$ where $x_0 = o, x_n = x$ and $x_k \leq x_{k+1}$.
- Extend these to measures $\mathbb{P}_n$ on $\mathcal{M}_n$ by using the fact that $\{B_x\}_{x \in T_n}$ is a finite partition of $\mathcal{M}_n$ and $\sum_{y \sim x} q(x, y) = 1$.
- Finally using an extension theorem we deduce the existence of a probability measure $\mathbb{P}$ on $\mathcal{M}$ that satisfies $\mathbb{P}\big|_{\mathcal{M}_n} = \mathbb{P}_n$.

Harmonic functions and Martingales

Functions f on the tree can be "lifted" to the boundary, by defining a sequence $\omega_n(f) : \partial T \to \mathbb{C}$, $\omega_n(f)(e) = f(e_n)$
This sequence satisfies the following:

- $\omega_n(f)$ is constant on each B_x for $x \in T_n$ therefore $\omega_n(f)$ is $\mathcal{M}_n$-measurable, hence $\mathcal{M}$-measurable.
- If f is Q-harmonic, then $\{\omega_n(f)\}_{n=1}^{\infty}$ is a martingale, i.e. $\mathbb{E}[\omega_{n+1}(f)|\mathcal{M}_n] = \omega_n(f), \quad \forall n \in \mathbb{N}$.

Similarly, a function h on the boundary that is M_n measurable, defines a function $\pi_n(h)$ on T_n by setting $\pi_n(h)(x) = h(B_x)$, therefore a sequence h_n of $\mathcal{M}_n$-measurable functions, defines a function on the whole tree. If h_n forms a martingale, the corresponding function on the tree is harmonic.

Harmonic functions on the tree are in one to one correspondence with martingales on the boundary.

Remark 1

The sequence $\omega_n(f)$ takes on the sets $B_x, x \in T_n$ the same values as f does on T_n. In that sense $\omega_n(f)$ can be seen as the restriction of f on the circle of radius n.

Remark 1

The sequence $\omega_n(f)$ takes on the sets $B_x, x \in T_n$ the same values as f does on T_n. In that sense $\omega_n(f)$ can be seen as the restriction of f on the circle of radius n.

On the space of $\mathcal{M}$-measurable functions we consider the topology of convergence in measure. This topology can be given by the following metric:

$$d(h,g) = \int_{\partial T} \frac{|h-g|}{1+|h-g|}\, d\mathbb{P}.$$

Definition

A function $f \in H_Q(T)$ is said to be *universal* if for every $\mathcal{M}$-measurable function h on ∂T there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that $\{\omega_{\lambda_n}(f)\}_{n=1}^{\infty}$ converges to h in measure. Their set is denoted by $\mathcal{U}(T)$.

Definition

A function $f \in H_Q(T)$ is said to be *universal* if for every $\mathcal{M}$-measurable function h on ∂T there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that $\{\omega_{\lambda_n}(f)\}_{n=1}^{\infty}$ converges to h in measure. Their set is denoted by $\mathcal{U}(T)$.

Remark 2

Since convergence in measure implies a.e. convergence of a subsequence, and since a.e. convergence implies convergence in measure, we have that the above definition is equivalent to demanding that a subsequence of $\omega_n(f)$ converges to h a.e.

First Results

Theorem 1, Abakoumov, Nestoridis, Picardello

The class $\mathcal{U}(T)$ is G_δ and dense in $H_Q(T)$.

The proof relies on a Baire Category Theorem argument.
Step 1: $\mathcal{U}(T)$ is a G_δ set.

There exists a dense sequence $\{h_j\}_{j=1}^{\infty}$, of functions defined on the boundary, such that each h_j is $\mathcal{M}_{n(j)}$-measurable, for some $n(j)$. Define the sets: $E(n,j,s) = \{f \in H_Q(T) \,|\, d(\omega_n(f), h_j) < \frac{1}{s}\}$ where: $n = 0, 1, \ldots j, s = 1, 2, \ldots$ and d is the metric inducing the topology of convergence in measure. Next it can be seen that:

$$\mathcal{U}(T) = \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=1}^{\infty} E(n,j,s).$$

First Results

The sets $\bigcup_{n=1}^{\infty} E(n,j,s)$ are open for every j,s, and as such, the set $\mathcal{U}(T)$ is a G_δ set.

Step 2: The sets $\bigcup_{n=1}^{\infty} E(n,j,s)$ are dense in $H_Q(T)$ for every j,s.

This is the most technical part of the proof. The idea is the following. Fix a harmonic function φ that we wish to approximate.

- To approximate φ appropriately, define $f=\varphi$ up to some level T_N, for N large enough.

First Results

The sets $\bigcup_{n=1}^{\infty} E(n,j,s)$ are open for every j,s, and as such, the set $\mathcal{U}(T)$ is a G_δ set.

Step 2: The sets $\bigcup_{n=1}^{\infty} E(n,j,s)$ are dense in $H_Q(T)$ for every j,s.

This is the most technical part of the proof. The idea is the following. Fix a harmonic function φ that we wish to approximate.

- To approximate φ appropriately, define $f=\varphi$ up to some level T_N, for N large enough.
- Extend f further, until some level T_{N+K} in such a way that it stays harmonic, and such that $f \in E(n,j,s)$ for some $n \geq 1$.

First Results

The sets $\bigcup_{n=1}^{\infty} E(n,j,s)$ are open for every j,s, and as such, the set $\mathcal{U}(T)$ is a G_δ set.

Step 2: The sets $\bigcup_{n=1}^{\infty} E(n,j,s)$ are dense in $H_Q(T)$ for every j,s.

This is the most technical part of the proof. The idea is the following. Fix a harmonic function φ that we wish to approximate.

- To approximate φ appropriately, define $f = \varphi$ up to some level T_N, for N large enough.
- Extend f further, until some level T_{N+K} in such a way that it stays harmonic, and such that $f \in E(n,j,s)$ for some $n \geq 1$.
- Define f on the rest of the tree by setting: $f(z) = f(x) \quad \forall z \geq x,\ x \in T_{N+K}$.

Frequently Universal Harmonic Functions

We will continue with the study of Frequently universal harmonic functions.

Definition

The *lower density* of a $A \subset \{0, 1, \dots\}$ is defined as:

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{card(\{m \in A \,|\, m \leq n\})}{n+1}$$

Similarly we define the *upper density* of a set as the $\limsup$ of the same quantity.

Definition

A function $f \in H_Q(T)$ is called *frequently universal* if for every non-void, open set V, contained in the set of $\mathcal{M}$-measurable functions on the boundary, the set $\{n \in \mathbb{N} \,|\, \omega_n(f) \in V\}$ has strictly positive lower density. The set of all frequently universal functions is denoted by $\mathcal{U}_{FM}(T)$.

Frequently Universal Harmonic Functions

Theorem 2, Abakoumov, Nestoridis, Picardello

The set $\mathcal{U}_{FM}(T)$ is dense and meager in $H_Q(T)$, i.e. it is dense and disjoint from a set that is G_δ-dense.

The main part of this proof is about constructing a frequently universal function. The rest follows by standard arguments.

In order to do so, we start by picking a dense sequence $\{h_n\}_{n=1}^{\infty}$ where each h_n is $\mathcal{M}_n$-measurable. Next we define a certain sequence ℓ_n, as follows:

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, \ldots$$

Frequently Universal Harmonic Functions

That sequence is obtained by taking the exponent of the largest power of 2 that divides the number n and adding 1. We also need the sequence $r_n = \sum_{k=1}^{n} \ell_k$. The sequence ℓ_n visits every natural number frequently (with a strictly positive lower density).

To construct a frequently harmonic function, we define a function f in such a way that $\omega_{r_n}(f) \in B(h_{\ell_n}, \frac{1}{2^{\ell_n}})$, where $B(h, r)$ is the open ball centred at h, of radius r. This is done inductively, following the same extension arguments as in Theorem 1.

Dense Subspaces

The next step is to talk about a property named *Algebraic genericity*. In the case of Universal martingales we have the following result:

Theorem 3, Abakoumov, Nestoridis, Picardello

The set $\mathcal{U}(T) \cup \{0\}$ contains a dense vector space.

The proof of this theorem relies on the fact that *Theorem 1*, holds in a slightly more general setting. It is possible to construct universal functions such that the subsequences ω_{λ_n} can all be chosen in a way that all λ_n are contained in a fixed, infinite, $\tau \subset \mathbb{N}$.

Dense Subspaces

Theorem 4, Biehler, Nestoridis, Stavrianidi

The set $\mathcal{U}_{FM}(T) \cup \{0\}$ contains a dense vector space.

There is no direct way to prove that the sum of two frequently universal functions is also frequently universal. Solution: Go to higher dimension!

We can consider the space $H_Q(T, \mathbb{C}^{\mathbb{N}})$ of harmonic functions on the tree, with values in $\mathbb{C}^{\mathbb{N}}$. By considering a frequently universal functions in $H_Q(T, \mathbb{C}^{\mathbb{N}})$ it is possible to prove that for a frequently universal function $f = (f_1, f_2, \ldots, f_n, \ldots)$, each of the f_n is frequently universal in $H_Q(T, \mathbb{C})$, and more importantly, $span\{f_n \,|\, n \in \mathbb{N}\}$ is contained in $H_Q(T, \mathbb{C})$

Dense Subspaces

In fact it is preferable to replace the space $\mathbb{C}^{\mathbb{N}}$ with an arbitrary separable, Fréchet space E and work with functions that take calues in $\mathbb{C}^{E}$. This space will have all the nice properties that enable the previous proofs to work. After establishing the necessary results, like Theorems 1,2,3 for this new space, we can proceed as mentioned above.

The final part of the proof boils down to finding a frequently universal function $f = (f_1, f_2, \dots, f_n, \dots)$ such that $\{f_n\}_{n=1}^{\infty}$ is dense in H_Q.

Combining these results, we obtain a dense linear space of frequently universal harmonic functions.

References

E. Abakumov, V. Nestoridis, M. Picardello, Frequently dense harmonic functions and universal martingales on trees Proc. Amer. Math. Soc. 149 (2021), 1905-1918

E. Abakumov, V. Nestoridis, M. Picardello, Universal properties of harmonic functions on trees, J. Math. Anal. Appl. 445(2) (2017), p.p. 1181-1187.

N. Biehler, V. Nestoridis, A. Stavrianidi, Algebraic genericity of frequently universal harmonic functions on trees, J. Math. Anal. Appl., V 489, 2020, 1, 124132, arXiv:1908.09767

N. Biehler, E.Nestoridi V. Nestoridis, Generalized harmonic functions on trees: Universality and frequent universality, J. Math. Anal. Appl., V 503, 2021, 1, 125277, arXiv: 2010.02149

Thank you for your attention!