

Extreme points of the unit ball in Lipschitz-free spaces

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Lipschitz spaces

Let (M, d) be a complete metric space. Fix a base point $0 \in M$.
The **Lipschitz constant** of $f: M \rightarrow \mathbb{R}$ is

$$\|f\|_L := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x \neq y \in M \right\}$$

The Lipschitz space

$$\text{Lip}_0(M) = \{f: M \rightarrow \mathbb{R} : \|f\|_L < \infty, f(0) = 0\}$$

is a Banach space with norm $\|\cdot\|_L$.

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$$\delta(x) : f \mapsto f(x)$$

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Theorem (Arens, Eells 1956)

$\mathcal{F}(M)^* \cong \text{Lip}_0(M)$, and the weak* topology on $B_{\text{Lip}_0(M)}$ is the topology of pointwise convergence.

Examples

- $\mathcal{F}(\mathbb{N}) \equiv \ell_1$:

$$\begin{aligned} T: \mathcal{F}(\mathbb{N}) &\rightarrow \ell_1 \\ \delta(n) &\mapsto (1, \dots, 1, 0, 0, \dots) \end{aligned}$$

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- $\mathcal{F}([0, 1]) \equiv L_1[0, 1]$:

$$\begin{aligned} T: \mathcal{F}([0, 1]) &\rightarrow L_1[0, 1] \\ \delta(x) &\mapsto \chi_{[0,x]} \end{aligned}$$

- $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$ similarly

Universal property

Theorem (Kadets 1985, Pestov 1986, Weaver 1999)

Let M be a metric space, X be a Banach space.

Let $f: M \rightarrow X$ be a **Lipschitz mapping** such that $f(0) = 0$.

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Then there is a **linear operator** $F: \mathcal{F}(M) \rightarrow X$ with $F|_{\delta(M)} = f$ and $\|F\| = \|f\|_L$.

$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ \downarrow \delta & \nearrow F & \\ \mathcal{F}(M) & & \end{array}$$

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In other words, $\text{Lip}_0(M, X) \equiv \mathcal{L}(\mathcal{F}(M), X)$.

Universal property

Corollary (Godefroy, Kalton 2003)

Let M, N be metric spaces.

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Introduction
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Extreme points
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Extreme molecules
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Motivation

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The universal property turns non-linear geometric problems into linear problems:

- If M and N are bi-Lipschitz equivalent then $\mathcal{F}(M) \sim \mathcal{F}(N)$.
- If N Lipschitz embeds into M then $\mathcal{F}(N) \hookrightarrow \mathcal{F}(M)$.
- If $N \subset M$ then $\mathcal{F}(N) \subset \mathcal{F}(M)$ isometrically.

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Example / Open problem

Is $\mathcal{F}(\ell_1)$ complemented in $\mathcal{F}(\ell_1)^{**}$?

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Is $\mathcal{F}(\ell_1)$ complemented in $\mathcal{F}(\ell_1)^{**}$?

If YES then ℓ_1 is determined by its Lipschitz structure
(i.e. if X is bi-Lipschitz equivalent to ℓ_1 then $X \sim \ell_1$).

Intersections and supports

Intersection Theorem (Aliaga, Pernecká 2020)

Let $K_i \subset M$ be closed subsets containing 0. Then

$$\bigcap_i \mathcal{F}(K_i) = \mathcal{F} \left(\bigcap_i K_i \right)$$

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Definition: For $m \in \mathcal{F}(M)$, the **support** of m is

$$\begin{aligned} \text{supp}(m) &= \bigcap \{K \subset M \text{ closed} : m \in \mathcal{F}(K \cup \{0\})\} \\ &= \bigcap \{K \subset M \text{ closed} : \langle m, f \rangle = 0 \text{ if } f|_K = 0\} \end{aligned}$$

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Intersection Theorem (equivalent form)

Let $m \in \mathcal{F}(M)$, $K \subset M$ closed. TFAE:

- $\text{supp}(m) \subset K$
- $m \in \mathcal{F}(K)$

Thus $m \in \mathcal{F}(\text{supp}(m))$, i.e. $f|_{\text{supp}(m)} = 0$ implies $\langle m, f \rangle = 0$.

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- x is a **preserved extreme point** of B_X iff

$$x = \frac{1}{2}(y^{**} + z^{**}), \quad y^{**}, z^{**} \in B_{X^{**}} \quad \implies \quad y^{**} = z^{**} = x$$

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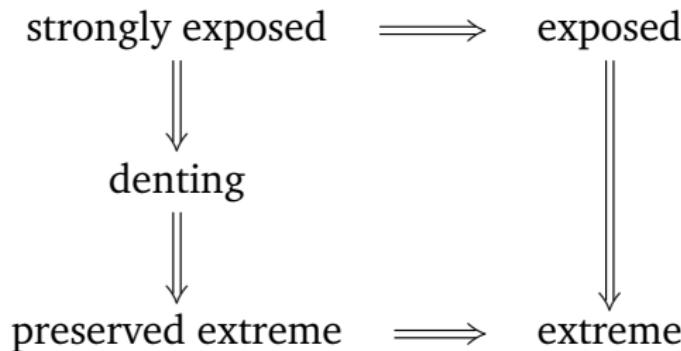
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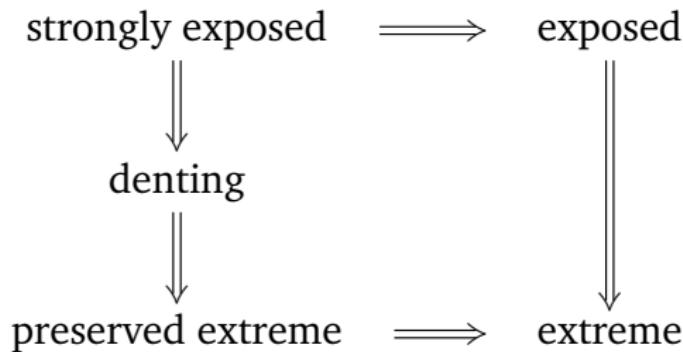
- x is a **strongly exposed point** of B_X iff there is $f \in B_{X^*}$ such that $\langle x, f \rangle = 1$ and

$$(y_n) \subset B_X \text{ and } \langle y_n, f \rangle \rightarrow 1 \implies y_n \rightarrow x$$

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Problem

Characterize all of them in Lipschitz-free spaces.

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where $\mathbf{m}_{xy} = \frac{\delta(x) - \delta(y)}{d(x,y)}$ is an **(elementary) molecule**.

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For any M , all extreme points are molecules.

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The **de Leeuw map** is the mapping $\Phi: \text{Lip}_0(M) \rightarrow C(\tilde{M})$

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- For each $\phi \in \text{Lip}_0(M)^*$ there is $\mu \in \mathcal{M}(\beta\tilde{M})$ with $\Phi^*\mu = \phi$

$$\langle f, \phi \rangle = \langle f, \Phi^*\mu \rangle = \int_{\beta\tilde{M}} (\Phi f) d\mu \quad \text{for } f \in \text{Lip}_0(M)$$

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and we may choose it so that $\|\mu\| = \|\phi\|$.

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We define the coordinate projections in $\beta\tilde{M}$ by extending

$$\begin{aligned} \mathfrak{p}: \tilde{M} &\rightarrow M \times M \\ (x, y) &\mapsto (x, y) \end{aligned} \implies \mathfrak{p}: \beta\tilde{M} \rightarrow \beta M \times \beta M$$

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- By weak* density, any μ can be replaced by a *positive* μ' with $\|\mu'\| = \|\mu\|$ and $\Phi^* \mu' = \Phi^* \mu$.
- If $\Phi^* \mu \in \mathcal{F}(M)$ then

$$\text{supp}(\Phi^* \mu) \subset \mathfrak{p}_1(\text{supp}(\mu)) \cup \mathfrak{p}_2(\text{supp}(\mu))$$

Extreme points in the bidual

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Extreme points in the bidual

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Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a molecule.

Corollary

Every extreme point of $B_{\mathcal{F}(M)}$ with finite support is a molecule.

Proof: $\mathcal{F}(S)^{**} = \mathcal{F}(S)$ if S is finite.

de Leeuw representations

For which $\mu \in \mathcal{M}(\beta\tilde{M})$ do we get $\Phi^*\mu \in \mathcal{F}(M)$?

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Proof for separable M :

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True for **sequences** by Lebesgue's DCT.

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Proof for nonseparable M :

Approximate μ by $\mu|_K$ for compact $K \subset \tilde{M}$:

$K \subset \tilde{S}$ where $S = p_1(K) \cup p_2(K) \subset M$ is compact.

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Is the converse true?

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If $m \in \mathcal{F}(M)$, is there always $\mu \in \mathcal{M}(\tilde{M})$ such that $\Phi^*\mu = m$?

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As stated: Yes. For every $\varepsilon > 0$ we may write

$$m = \sum_{n=1}^{\infty} a_n m_{x_n y_n} = \Phi^* \left(\sum_{n=1}^{\infty} a_n \delta_{(x_n, y_n)} \right)$$

with $\sum |a_n| < \|m\| + \varepsilon$.

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If $m \in \mathcal{F}(M)$, is there always $\mu \in \mathcal{M}(\tilde{M})$ such that $\Phi^*\mu = m$
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If $m \in \mathcal{F}(M)$, is there always $\mu \in \mathcal{M}(\tilde{M})$ such that $\Phi^*\mu = m$ and moreover $\|\mu\| = \|m\|$?

Yes if M compact and purely 1-unrectifiable (i.e. it contains no bi-Lipschitz copy of a subset of \mathbb{R} with positive measure).

Optimal positive representations

$\mu \in \mathcal{M}(\beta\tilde{M})$ is an **optimal positive representation (OPR)** of $\phi \in \text{Lip}_0(M)^*$ if

$$\Phi^*\mu = \phi \quad , \quad \mu \geq 0 \quad , \quad \|\mu\| = \|\phi\|$$

Every $\phi \in \text{Lip}_0(M)^*$ has at least one OPR.

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Fact

If $\mu \in \mathcal{M}(\beta\tilde{M})$ is an OPR, then

- any $c \cdot \mu$ for $c \geq 0$
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are also OPRs.

Proof: $\|\Phi^*\mu\| \leq \|\Phi^*\lambda\| + \|\Phi^*(\mu - \lambda)\| \leq \underbrace{\|\lambda\| + \|\mu - \lambda\|}_{\text{by positivity}} = \|\mu\|$

Representation of extreme points

Proposition

Let $m \in \text{ext } B_{\mathcal{F}(M)}$ and fix an OPR μ of m .

Suppose that $0 \leq \lambda \leq \mu$ and $\Phi^* \lambda \in \mathcal{F}(M)$. Then $\Phi^* \lambda = \|\lambda\| \cdot m$.

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$$m = \Phi^*\lambda + \Phi^*(\mu - \lambda) = \|\lambda\| \Phi^* \left(\frac{\lambda}{\|\lambda\|} \right) + \|\mu - \lambda\| \Phi^* \left(\frac{\mu - \lambda}{\|\mu - \lambda\|} \right)$$

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$$\text{Therefore } m = \Phi^* \left(\frac{\lambda}{\|\lambda\|} \right) = \Phi^* \left(\frac{\mu - \lambda}{\|\mu - \lambda\|} \right).$$

Support reduction argument

Theorem

Let $m \in \text{ext } B_{\mathcal{F}(M)}$. Suppose that m has an OPR that is concentrated on \tilde{M} . Then m is a molecule.

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Let $\lambda_r = \mu|_{\tilde{E}_r}$. Then $0 \leq \lambda_r \leq \mu$ and $\Phi^* \lambda_r \in \mathcal{F}(M)$.
(because λ_r is concentrated on \tilde{M})

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It follows that $\text{supp}(m) \subset \bigcap_{r>0} E_r = \{x, y\}$.

Compact case

Theorem (Aliaga 2021)

Let $m \in \text{ext } B_{\mathcal{F}(M)}$. If M is compact then m is a molecule.

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Construct a Lipschitz h with $h = 1$ on E_r and $h = 0$ outside E_{2r} .

Define $\lambda_r \in \mathcal{M}(\tilde{\beta M})$ as

$$d\lambda_r(x, y) = h(x) \cdot h(y) \cdot d\mu(x, y) \quad , \quad (x, y) \in \tilde{M}$$

If we prove $\Phi^* \lambda_r \in \mathcal{F}(M)$, then $\text{supp}(m) \subset E_{2r}$ as before.

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Lemma

Suppose that M is compact, $\mu \in \mathcal{M}(\beta\tilde{M})$ and $\Phi^*\mu \in \mathcal{F}(M)$.
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$$\Phi(fh)(x,y) = \Phi(f)(x,y) \cdot h(x) + \Phi(h)(x,y) \cdot f(y)$$

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$$\underbrace{\langle \Phi^*\mu, fh \rangle}_{\text{weighted functional, } \in \mathcal{F}(M)} = \langle f, \Phi^*\lambda \rangle + \underbrace{\int_{\beta M} f d((\mathfrak{p}_2)_\sharp \nu)}_{\in \mathcal{F}(M) \text{ iff } \beta M = M}$$

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Proof:

$$\begin{aligned}\Phi(fh)(x, y) &= \Phi(f)(x, y) \cdot h(x) + \Phi(h)(x, y) \cdot f(y) \\ \Phi(fh) &= \Phi(f) \cdot (h \circ \mathfrak{p}_1) + \Phi(h) \cdot (f \circ \mathfrak{p}_2)\end{aligned}$$

$$\begin{aligned}\int_{\beta\tilde{M}} \Phi(fh) d\mu &= \int_{\beta\tilde{M}} (\Phi f) \cdot (h \circ \mathfrak{p}_1) d\mu + \int_{\beta\tilde{M}} (\Phi h) \cdot (f \circ \mathfrak{p}_2) d\mu \\ \underbrace{\langle \Phi^*\mu, fh \rangle}_{\text{weighted functional, } \in \mathcal{F}(M)} &= \langle f, \Phi^*\lambda \rangle + \underbrace{\int_M f d((\mathfrak{p}_2)_\sharp \nu)}_{\in \mathcal{F}(M) \text{ iff } \beta M = M}\end{aligned}$$

Necessary conditions

Let $p, q \in M$. The **metric segment** between p and q is

$$[p, q] = \{x \in M : d(p, x) + d(q, x) = d(p, q)\}$$

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If m_{pq} is an extreme point of $B_{\mathcal{F}(M)}$, then $[p, q] = \{p, q\}$.

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Proposition

If m_{pq} is an extreme point of $B_{\mathcal{F}(M)}$, then $[p, q] = \{p, q\}$.

Proof: If $x \in [p, q]$ then $m_{pq} \in [m_{px}, m_{xq}]$:

$$\begin{aligned} m_{pq} &= \frac{\delta(p) - \delta(q)}{d(p, q)} = \frac{\delta(p) - \delta(x)}{d(p, q)} + \frac{\delta(x) - \delta(q)}{d(p, q)} \\ &= \frac{d(p, x)}{d(p, q)} m_{px} + \frac{d(x, q)}{d(p, q)} m_{xq}. \end{aligned}$$

Necessary conditions

Let $p, q \in M$. The **extended metric segment** between p and q is

$$[p, q]_\beta = \{\xi \in \beta M : d(p, \xi) + d(q, \xi) = d(p, q)\}$$

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Let $p, q \in M$. The **extended metric segment** between p and q is

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Proposition (Aliaga, Guirao 2019)

If m_{pq} is preserved extreme in $B_{\mathcal{F}(M)}$, then $[p, q]_\beta = \{p, q\}$.

Necessary conditions

Let $p, q \in M$. The **extended metric segment** between p and q is

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Proposition (Aliaga, Guirao 2019)

If m_{pq} is preserved extreme in $B_{\mathcal{F}(M)}$, then $[p, q]_\beta = \{p, q\}$.

Proof: If $\xi \in [p, q]_\beta$ then $\delta(\xi) \in \text{Lip}_0(M)^* = \mathcal{F}(M)^{**}$ and

$$m_{pq} = \frac{d(p, \xi)}{d(p, q)} m_{p\xi} + \frac{d(q, \xi)}{d(p, q)} m_{\xi q}$$

where $m_{p\xi} = \frac{\delta(p) - \delta(\xi)}{d(p, \xi)}$, $m_{\xi q} = \frac{\delta(\xi) - \delta(q)}{d(q, \xi)} \in S_{\mathcal{F}(M)^{**}}$.

...are also sufficient conditions

Theorem (Aliaga, Guirao 2019)

If $[p, q]_\beta = \{p, q\}$ then m_{pq} is a preserved extreme point of $B_{\mathcal{F}(M)}$.

Theorem (Aliaga, Pernecká 2020)

If $[p, q] = \{p, q\}$ then m_{pq} is an extreme point of $B_{\mathcal{F}(M)}$.

Thank you for your attention!

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