Functional calculus for submarkovian semigroups on weighted L^2 spaces

Christoph Kriegler (Clermont-Ferrand, France), joint work with Komla Domelevo and Stefanie Petermichl (Würzburg, Germany)

GDR AFHP Besançon, France – September 2021

c_0-semigroups

Definition: c_0-semigroups

Let X be a Banach space. Let $(T_t)_{t\geq 0}$ be a family of bounded linear operators $X \to X$. Then $(T_t)_{t\geq 0}$ is called a c_0-semigroup if

1. $T_0 = \mathrm{Id}_X$,
2. $T_{t+s} = T_t \circ T_s$ for any $t, s \geq 0$,
3. $T_t x \to x$ as $t \to 0+$ for any $x \in X$.

Fact:

Any c_0-semigroup is uniquely determined by its generator A, where

$$Ax = \lim_{t\to 0+} \frac{1}{t}(\mathrm{Id}_X - T_t)x$$

with domain $D(A) = \{x \in X : \text{ the above limit exists}\}$.
A is always closed and densely defined.

From Fourier multipliers to spectral multipliers

E.g. $A = -\Delta$ on $X = L^p(\mathbb{R}^d)$ for some $1 < p < \infty$. Then $(T_t)_t$ is the classical **heat semigroup.**
For $m : (0, \infty) \to \mathbb{C}$, have operator $m(A) = m(-\Delta)$, Fourier multiplier with symbol $m(|\xi|^2)$.
In particular, if $m_t(\lambda) = e^{-t\lambda}$, then $m_t(-\Delta) = T_t$, i.e. one recovers the semigroup.
Other semigroups? How to define $m(A)$?
If for $m : (0, \infty) \to \mathbb{C}$ there exists $\beta : (0, \infty) \to \mathbb{C}$ such that

$$m(\lambda) = \int_0^\infty \beta(t) \lambda e^{-\lambda t} dt \quad (\lambda > 0)$$

then formally

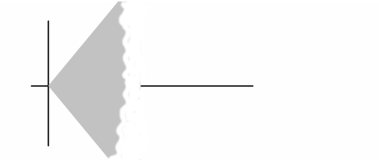
$$m(A) = \int_0^\infty \beta(t) A T_t dt.$$

The H^∞ class

Definition: Sector and H^∞ class

Let $\omega \in (0, \pi)$ be an angle. Define the sector

$$\Sigma_\omega := \{\lambda \in \mathbb{C}\setminus\{0\} : \ |\arg \lambda| < \omega\}$$



Define moreover

$$H^\infty(\Sigma_\omega) = \Big\{m : \Sigma_\omega \to \mathbb{C} : \ m \text{ analytic and} \\ \|m\|_{\infty,\omega} := \sup_{\lambda\in\Sigma_\omega} |m(\lambda)| < \infty\Big\}.$$

The class $m \in H^\infty(\Sigma_\omega)$ is often appropriate to define $m(A) \in B(X)$.

Construction of the H^∞ calculus

Fact [Cowling Doust McIntosh Yagi 1996]

Let $\theta > \frac{\pi}{2}$ and $m \in H^\infty(\Sigma_\theta)$. Then there does exist $\beta \in L^\infty(\mathbb{R}_+)$ such that $\|\beta\|_{L^\infty(\mathbb{R}_+)} \leq C\,\|m\|_{\infty,\theta}$ and

$$m(\lambda) = \int_0^\infty \beta(t)\lambda e^{-\lambda t}dt \quad (\lambda > 0).$$

So if

$$\int_0^\infty |\langle AT_t f, g\rangle| dt \leq C\,\|f\|_X\,\|g\|_{X^*}\,,$$

then for

$$\langle m(A)f, g\rangle := \int_0^\infty \beta(t)\langle AT_t f, g\rangle dt,$$

we have

$$|\langle m(A)f, g\rangle| \leq C\,\|\beta\|_\infty\,\|f\|_X\,\|g\|_{X^*} \leq C'\,\|m\|_{\infty,\theta}\,\|f\|_X\,\|g\|_{X^*}\,.$$

Thus, $m(A)$ defines a bounded operator on X for any $m \in H^\infty(\Sigma_\theta)$.

The H^∞ calculus

Definition: H^∞ calculus

Let A be a semigroup generator and $\theta \in (0, \pi)$. Then A has a (bounded) $H^\infty(\Sigma_\theta)$ calculus if

$$\|m(A)\|_{B(X)} \leq C \|m\|_{\infty,\theta}$$

for any $m \in H^\infty(\Sigma_\theta)$.

If θ becomes **smaller**, then the $H^\infty(\Sigma_\theta)$ calculus becomes a **stronger statement**.

If $\theta < \frac{\pi}{2}$ and $z \in \Sigma_{\frac{\pi}{2}-\theta}$, then $m_z : \lambda \mapsto e^{-z\lambda} \in H^\infty(\Sigma_\theta)$.

Thus if A has $H^\infty(\Sigma_\theta)$ calculus, then $T_z = m_z(A) = e^{-zA}$ is a well-defined **analytic semigroup**.

Weak square function for smaller angles

Question: How to obtain $H^\infty(\Sigma_\theta)$ calculus for smaller (i.e. better) angles $\theta < \frac{\pi}{2}$?

Proposition [Cowling Doust McIntosh Yagi 1996]

Let $\theta \in (0, \frac{\pi}{2})$ and $\phi \in (\frac{\pi}{2} - \theta, \frac{\pi}{2})$.
If $(T_z)_{z \in \Sigma_\phi}$ is an analytic semigroup and

$$\int_0^\infty |\langle A T_{e^{\pm i\phi} t} f, g \rangle| \, dt \leq C \, \|f\|_X \, \|g\|_{X^*} ,$$

then A has a bounded $H^\infty(\Sigma_\theta)$ calculus.

Consequences of H^∞ calculus

1. A has $H^\infty(\Sigma_\theta)$ calculus on $X = L^p$-space $\implies$ Paley-Littlewood decomposition

$$\|x\|_p \cong \left\| \left(\sum_{n\in\mathbb{Z}} |\psi(2^n A)x|^2 \right)^{\frac{1}{2}} \right\|_p .$$

2. A has $H^\infty(\Sigma_\theta)$ calculus for $\theta < \frac{\pi}{2}$ and $X = L^p$-space $\implies$ the evolution equation associated with A has maximal regularity:

$$\begin{cases} \frac{\partial}{\partial t} y(t) + Ay(t) & = f(t) \\ y(0) & = 0 \end{cases} .$$

Classes of c_0-semigroups

Definition: (Sub)markovian semigroups

Let (Ω, μ) be a σ-finite measure space. Let $(T_t)_{t\geq 0}$ be a c_0-semigroup on $L^2(\Omega)$. Consider the conditions

1. T_t is self-adjoint on $L^2(\Omega)$ for any $t \geq 0$.
2. $\|T_t\|_{p\to p} \leq 1$ for any $t \geq 0$ and any $1 \leq p \leq \infty$.
3. $T_t(f) \geq 0$ for any $f \in L^2(\Omega)$ such that $f \geq 0$.
4. $T_t(1) = 1$.

(1)-(2): semigroup of **symmetric contractions**. In this case, have contractive c_0-semigroup $(T_t)_t$ acting on $L^p(\Omega)$, $1 \leq p < \infty$.
(1)-(3): **submarkovian semigroup**.
(1)-(4): **markovian semigroup**.

H^∞ functional calculus

Theorem [Stein 1970, Cowling 1983, Meda 1990]

Let $1 < p < \infty$.
Let $(T_t)_t$ be a semigroup of symmetric contractions acting on $L^p(\Omega)$.
Let $\theta > \pi \left| \frac{1}{p} - \frac{1}{2} \right|$.
Then A has an $H^\infty(\Sigma_\theta)$ calculus.

Optimal angle of H^∞ functional calculus

Theorem [Carbonaro-Dragičević 2017]

Let $1 < p < \infty$.
Let $(T_t)_t$ be a semigroup of symmetric contractions acting on $L^p(\Omega)$.
Let $\theta > \theta_p = \arcsin \left|1 - \frac{2}{p}\right|$.
Then A has $H^\infty(\Sigma_\theta)$ calculus.
The angle θ_p is essentially optimal: If $(T_t)_t$ is the Ornstein-Uhlenbeck semigroup, then false for any $\theta < \theta_p$.

Proof of Carbonaro-Dragičević's Theorem

Elements of proof: By [Cowling Doust McIntosh Yagi], it suffices to estimate for angle $|\phi| < \frac{\pi}{2} - \theta_p$,

$$\int_0^\infty |\langle AT_{te^{i\phi}} f, T_{te^{-i\phi}} g \rangle| \, dt \leq C \|f\|_p \|g\|_{p^*} \quad (f \in L^p(\Omega), g \in L^{p^*}(\Omega)).$$

Introduce the functional $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathcal{E}(t) = \int_\Omega B(T_{te^{i\phi}}(f)(x), T_{te^{-i\phi}}(g)(x)) d\mu(x),$$

where $B : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ determined later.

Proof of optimal H^∞ calculus continued

Want to put $-\mathcal{E}'(t)$ in between the above weak square function estimate. Have

$$-\mathcal{E}'(t) = \Re \int_\Omega e^{i\phi}(AT_{te^{i\phi}}f)\partial_1 B(T_{te^{i\phi}}(f), T_{te^{-i\phi}}(g)) + e^{-i\phi}(AT_{te^{-i\phi}}g)\partial_2 B(T_{te^{i\phi}}(f), T_{te^{-i\phi}}(g))d\mu,$$

Lemma

The following are equivalent. There exists a function B such that …

1. $-\mathcal{E}'(t) \geq c_\phi \left|\langle AT_{te^{i\phi}}(f), T_{te^{-i\phi}}(g)\rangle\right|$ for any sgrp.
2. For $A = \mathcal{G} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ acting on $\Omega = \{a, b\}$, this inequality holds.
3. For $A = \mathcal{G}$ and $t = 0$, this inequality holds: $-\mathcal{E}'(0) \geq c_\phi \left|\langle \mathcal{G}f, g\rangle\right| = c_\phi \left|f(a) - f(b)\right| \cdot \left|g(a) - g(b)\right|$.

The last condition holds if B satisfies a certain **convexity**.

Proof of optimal H^∞ calculus continued

If we can find such a convex function B, then by the Lemma,

$$\begin{aligned}\int_0^\infty |\langle AT_{te^{i\phi}}(f), T_{te^{-i\phi}}(g)\rangle|\, dt &\lesssim -\int_0^\infty \mathcal{E}'(t)dt \\ &= \mathcal{E}(0) - \mathcal{E}(\infty) \\ &\le \int_\Omega B(T_0(f), T_0(g))d\mu - 0 \\ &\lesssim \int_\Omega |f|^p + |g|^{p^*} d\mu \\ &= \|f\|_p^p + \|g\|_{p^*}^{p^*}\end{aligned}$$

provided B **takes positive values and** $B(x,y) \le C(|x|^p + |y|^{p^*})$.

Proof of optimal H^∞ calculus continued

In all we need to find a function B depending on ϕ such that

- B satisfies a certain convexity (depending on ϕ).
- $0 \leq B(x, y) \leq C\left(|x|^p + |y|^{p^*}\right)$.
- B is everywhere C^1 and piecewise C^2.

Such a function is called **Bellman function** in view of similar functions for other problems in analysis. Carbonaro-Dragičević found existence of B with all these properties exactly when $|\phi| < \frac{\pi}{2} - \theta_p$. One deduces the weak square function estimate. □

Weighted L^p spaces

Now modify setting. Let (Ω, μ) be a measure space.
A measurable function $w : \Omega \to (0, \infty)$ is called a weight.
Have a weighted space $L^p(w) = L^p(\Omega, wd\mu)$ with
$\|f\|_{L^p(w)} = \left(\int_\Omega |f(x)|^p w(x) d\mu(x)\right)^{\frac{1}{p}}$.
Question: For which weights w and operators T,
$\|T\|_{L^p(w)\to L^p(w)} = \left\| M_{w^{\frac{1}{p}}} T M_{w^{-\frac{1}{p}}} \right\|_{L^p(\Omega,\mu)\to L^p(\Omega,\mu)} < \infty$?
Fact: If $\Omega = \mathbb{R}$ and T is the Hilbert transform (singular integral operator), then answer "yes" iff $[w]_{A_p} < \infty$, where

$$[w]_{A_p} = \sup_B \left(\frac{1}{|B|}\int_B wd\mu\right)\left(\frac{1}{|B|}\int_B w^{-\frac{p^*}{p}} d\mu\right)^{\frac{p}{p^*}}.$$

Semigroup weights

Question: What can we say if $T = m(A)$ stems from a semigroup? If $A = -\Delta$ on $\mathbb{R}^d$, then

$$[w]_{A_p} \cong \sup_{t>0} \sup_{x \in \mathbb{R}^d} T_t(w)(x) \left[T_t(w^{-\frac{p^*}{p}})(x) \right]^{\frac{p}{p^*}} =: Q_p^A(w).$$

Take the right hand side as definition of class of weights for a markovian semigroup $(T_t)_t$.

Theorem [Domelevo-K.-Petermichl 2021]

Let (Ω, μ) be a σ-finite measure space.
Let $(T_t)_t$ be a markovian semigroup.
Fix $p = 2$.
Assume some technical conditions.
Let w be a weight such that $Q_2^A(w) < \infty$.
Then A has a $H^\infty(\Sigma_\theta)$ calculus on $L^2(w)$ for any $\theta > \frac{\pi}{2}$.

Elements of proof

Follow Carbonaro-Dragičević's idea. There is no angle ϕ any more. Want

$$\int_0^\infty |\langle AT_t f, T_t g\rangle| dt \leq C \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})} .$$

Let $Q = Q_2^A(w) < \infty$. Put

$$\mathcal{E}(t) = \int_\Omega B_Q(T_t(f), T_t(g), T_t(w^{-1}), T_t(w)) d\mu$$

for some function
$B_Q : D(B_Q) = \mathbb{C} \times \mathbb{C} \times \{(w, v) \in \mathbb{R}_+^2 : 1 \leq wv \leq Q\} \to \mathbb{R}$ to find.

Proof continued

In order to put $-\mathcal{E}'(t)$ into the key inequality, need to find B_Q such that

- B_Q is defined on domain $D(B_Q)$ depending on Q.
- B_Q satisfies a *weak* convexity (difficulty: $D(B_Q)$ is not convex!).
- $0 \leq B_Q(x, y, w, v) \leq C\left(\frac{|x|^2}{w} + \frac{|y|^2}{v}\right)$.
- B_Q and its first derivative satisfy some technical conditions (difficulty: $\mathcal{E}(t)$ is not differentiable at $t = 0$).

$\square$

Variants

Variants of Theorem

1. There exists also a version for submarkovian semigroups with a modified weight characteristic.
2. Also get boundedness of $m(A)$ on $L^2(w)$ in case m holomorphic on $\mathbb{C}_+ = \Sigma_{\frac{\pi}{2}}$ plus regularity of m on boundary $= i\mathbb{R}$.
3. For certain semigroups, can lower the $H^\infty(\Sigma_\theta)$ calculus angle to some $\theta = \theta(w) < \frac{\pi}{2}$. Then have bounded semigroup $\|T_t\|_{L^2(w) \to L^2(w)} \leq C$ and maximal regularity.

Extensions: Smaller angle

Theorem [Duong-Sikora-Yan 2011, Gong-Yan 2014]

Let $(T_t)_t$ is a self-adjoint semigroup on $L^2(\mathbb{R}^d, dx)$ (or more generally on $L^2(\Omega, \mu)$ where (Ω, d, μ) is a space of homogeneous type), having an integral kernel p_t with Gaussian estimates.
Let $1 < p < \infty$ and $\theta \in (0, \pi)$ **(small)**.
Then [Domelevo K. Petermichl] holds, even on $L^p(\Omega, w d\mu)$.
Moreover, there is $s > 0$ such that

$$\|m(A)\|_{L^p(w) \to L^p(w)} \leq C\theta^{-s}(|m(0)| + \|m\|_{\infty,\theta}) \tag{1}$$
$$(\theta \in (0, \pi),\ m \in H^\infty(\Sigma_\theta))$$

Negative result on small angle

Theorem [Domelevo K. Petermichl 2021]

There exists a markovian semigroup $(T_t)_t$ without Gaussian estimates on some probability space (Ω, μ) and a weight w with $Q_2^A(w) < \infty$ such that for no $s > 0$, (1) holds with $p = 2$.

Thank you for your attention